Boundary Constructions for CR Manifolds and Fefferman Spaces

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Chapter 1

Introduction

In the times of the mechanical view of the world time and space were regarded as unassailable bastions. Everything happens in an unchangeable space. And the same was postulated for the time. It flows on and on unswayable, immutable. Causality prevents chaos to emerge. Then Lorentz - and later but more consequently Einstein - postulated the speed of light as the limit for all movements. Space and time are no longer seen as "super-quantities" but are influenced mainly by gravity. However how do space, matter, time and reality influence each other? Where is the boundary of space, the boundary of time? Are there such boundaries at all? How does a space ship commander knows, he is close to the boundary of the universe? If there are miscellaneous realities how are they delimited? While searching for ways to describe such "boundary phenomena" physicists developed several boundary construction for Lorentzian and conformal manifolds. In 1971 B.G.Schmidt ([Sch71]) proposed the b-boundary as a way to avoid the problems which occurred with former boundary constructions. In those days however the language of Cartan geometry was not as far advanced as it is nowadays which made it even harder to handle the construction and explicitly compute boundaries. Still several papers were published about the b-boundary such as [Cli66], [Sch71], ..., [HE73], [Cla78], [Bos79], [Dod79] [Joh79] to name a few. Later on physicists concentrated on causal and conformal boundaries. A nice overview on these works can be found in [San09] starting by first constructions in [GKP72] and [HE73] followed for example by [Har98] and [Har07] leading to [MR03], [FS08] and [San09] and many more. The b-boundary found solely sporadic attention for example in [Sta99] or [Fra08].

This is rather sad, since this boundary has the great advantage of being intrinsically defined in a natural way which can be applied to the broad variety of Cartan geometries. This makes it very interesting for mathematicians, as boundaries can be studied when searching for invariants, when classifying spaces. With the great progress in Cartan geometry and parabolic geometry during the last years (see for example [CS00], [Cap02], [CG02], [CS03], [Cap06], [CS09]) we have strong tools for studying the construction of Schmidt in a far more general and effective way. Schmidt chose the name b-boundary in order to indicate, that this boundary was constructed using a bundle. However since this is a very special bundle, namely the Cartan bundle, and since this construction has a far greater area of application than discussed by Schmidt we want to denote this boundary with ∂_{CB}, the Cartan boundary. We hope that, in keeping the original b as part of the denotation, this name will give credit to Cartan and Schmidt as well and further not cause confusion with conformal, causal, Cauchy and other boundaries which are often denoted with a ∂_c or in similar ways.

The aim of this thesis is to discuss the Cartan boundary of CR manifolds and their Fefferman spaces. CR manifolds, an odd dimensional correspondent of Kähler manifolds, are real smooth manifolds M^{2n+1} endowed with an involutive complex subbundle $T_{10} \subset TM^{\mathbb{C}}$ of dimension n such that $T_{10} \oplus \overline{T_{10}}$ is a direct sum.

Real hypersurfaces M of complex manifolds (N, J_N) are CR manifolds for example.

Strictly pseudo-convex CR manifolds (M, T_{10}, θ) are furthermore endowed with a pseudo-hermitian form $\theta \in \Omega^1(M^{2n+1})$, i.e. a form which vanishes nowhere but is zero if restricted to the subbundle $T_{10} \oplus \overline{T_{10}}$, such that the corresponding Levi form is positive definite. The Levi form is defined via

$$\begin{array}{rcl} L_\theta : \Gamma(T_{10}) \times \Gamma(T_{10}) & \longrightarrow & C^\infty(M, \mathbb{C}) \\ L_\theta(X, Y) & := & -i \cdot d\theta(X, \overline{Y}). \end{array}$$

Now according to [BL04] the Fefferman space of a strictly pseudo-convex CR manifold is defined as the S^1-principal bundle obtained by dividing the canonical complex line bundle ($\mathcal{K} := \left\{\omega \in \bigwedge^{n+1}(TM^{\mathbb{C}})^* \;\middle|\; i_V\omega = 0 \text{ for all } \; V \in \overline{T_{10}}\right\}$) without the zero by $\mathbb{R}^+$:

$$\mathcal{F} := \mathcal{K}^*/_{\mathbb{R}^+} \longrightarrow M.$$

The conformal class of the Fefferman space is obtained with the help of the Tanaka Webster connection, a covariant derivative $\nabla^W : \Gamma(T_{10}) \times \Gamma(TM^{\mathbb{C}}) \longrightarrow \Gamma(T_{10})$ uniquely defined by the following conditions

1. ∇^W preserves the Levi form L_θ, that is for all sections $X, Y, Z \in \Gamma(T_{10})$ we have
$$Z\big(L_\theta(X,Y)\big) = L_\theta\big(\nabla^W_Z X, Y\big) + L_\theta\big(X, \nabla^W_{\overline{Z}} Y\big),$$
2. $\nabla^W_T X = pr_{T_{10}}[T, X]$ for all $X \in \Gamma(T_{10})$ and
3. $\nabla^W_{\overline{Y}} X = pr_{T_{10}}[\overline{Y}, X]$ for all $X, Y \in \Gamma(T_{10})$.

We set
$$A^\theta := A^W - \frac{i}{2(n+1)} R^W \cdot \theta \in \mathcal{C}(\mathcal{F})$$
and the conformal class of the metric
$$h_\theta := \pi^* L_\theta - i\frac{4}{n+2}\pi^*\theta \odot A^\theta$$
is CR-invariant, i.e. $[h_\theta] = [h_{f\cdot\theta}]$.
$(F, [h_\theta])$ is called the Fefferman space of the strictly pseudo-convex CR manifold (M^{2m+1}, T_{10}).
To deal with the boundaries of the objects just described we have to set up their Cartan geometry according to [CS00]. Given a Lie group G with Lie algebra $LA(G) = \mathfrak{g}$ and a closed subgroup $P \subset G$ a Cartan geometry of type (G, P) is a P-principal bundle $\pi : \mathcal{G} \longrightarrow M$ endowed with a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ which is a P-equivariant one form on $\mathcal{G}$ with values in the Lie algebra $\mathfrak{g}$ such that the generators of the fundamental vector fields are reproduced and the tangent bundle $T\mathcal{G}$ is trivialized by ω. The curvature of the Cartan connection is denoted with Ω^ω.
A special class of Cartan geometries are the parabolic geometries. Here the Lie algebra $\mathfrak{g} = LA(G)$ is equipped with a $|k|$-grading, that is to say a splitting $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ holds for all $i, j = -k, \ldots, k$. Furthermore the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is supposed to be generated by $\mathfrak{g}_{-1}$ and no simple ideal of $\mathfrak{g}$ may be contained in $\mathfrak{g}_0$. The Lie algebra $\mathfrak{g}$ is then called an effective semisimple $|k|$-graded Lie algebra.
The name parabolic geometry is inspired by the fact that in the complex case those geometries are the ones with a parabolic subgroup P.
We have the following theorem ensuring the existence and uniqueness of Cartan bundles and Cartan connections especially for CR and conformal geometries.

Proposition 1.1 *Suppose that G is a semisimple Lie group whose Lie algebra $\mathfrak{g}$ is endowed with a $|k|$-grading, such that all cohomology groups $H^1_l(\mathfrak{g}_-, \mathfrak{g})$ with $l > 0$ are trivial. Furthermore let M be a smooth manifold endowed with a filtration of its tangent bundle $T^{-k}M = TM \supset T^{-k+1}M \supset \cdots \supset T^{-1}M$ by vector subbundles, such that for each $i = -k, \ldots, -1$ the rank of T^iM equals the dimension of $\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{-1}$.*

Then there is a bijective correspondence between isomorphism classes of reductions to the structure group G_0 of the associated graded vector bundle to the tangent bundle, which satisfy the structure equations, and isomorphism classes of P-principal bundles over M endowed with Cartan connections with ∂^*-closed curvature and $(\Omega^\omega)^l \equiv 0$ for all $l \leq 0$.

In Chapter 3 this theory is explained in the general case according to [CS00] in detail so that beginners can use this chapter to get familiar with the concepts and tools of Cartan and parabolic geometry. There will also be a side trip to tractor calculus, as far as we will need it later on. In Chapter 4 the general theory of Cartan geometry will be applied to the setting of CR-manifolds.
Using the tools of Cartan geometry there is another way of defining the Fefferman space of a CR manifold from [CG08] which we will discuss in detail in Chapter 5. This second approach leads to a strong relationship between the Cartan geometries of a CR manifold M and the corresponding Fefferman space $\mathcal{F} := \mathcal{G}/_{\widetilde{P}\cap G}$, namely

$$\widetilde{\mathcal{G}} = \mathcal{G} \times_{G\cap\widetilde{P}} \widetilde{P},$$

where the tilde denotes the objects of the Fefferman space. The Cartan connection of the Fefferman space will be given as

$$\widetilde{\omega}_{[u,\widetilde{p}]} = Ad(\widetilde{p}^{-1}) \circ \pi^*_{\mathcal{G}}\omega + \pi^*_{\widetilde{P}}\omega_{\widetilde{P}}.$$

However we want to point out, that in order to achieve this a rather strong assumption, the existence of an $(n+2)$nd root of the anticanonical complex line bundle, has to be made. This root exists for CR manifolds embedded in $\mathbb{C}^{n+1}$ globally and locally we have this root for any CR manifold. As we are interested in boundaries we need a global construction of the Fefferman space. So for boundary considerations it is very helpful to have both constructions at hand.
Considering both constructions discussed we find that local results on the Fefferman space can be transferred from one construction to the other since we have conformal coverings

$$\begin{tikzcd}
 & \mathcal{E}(1,0) \times_{[S^1,\lambda^{-1}]} S^1 \arrow[dl, "2:1"'] \arrow[dr, "(n+2):1"] & \\
\mathcal{F}_{[CG08]} & & \mathcal{F}_{[BL04]}.
\end{tikzcd}$$

Now in Chapter 6 the Cartan boundary can be defined according to [Sch71]. Since a Cartan connection gives a global trivialization of the Cartan bundle $\mathcal{G}$ it induces a global frame of $\mathcal{G}$ and hence a Riemannian metric ϱ. Now the Riemannian manifold $(\mathcal{G}, \varrho)$ can be completed by Cauchy completion and the obtained boundary is projected to define the Cartan boundary of the manifold itself, $\partial_{CB}M := \overline{\mathcal{G}}/_P \setminus M$.

$$\begin{array}{ccc}
\mathcal{G} & \longrightarrow & \overline{\mathcal{G}} \\
\pi\downarrow & & \downarrow\overline{\pi} \\
\mathcal{G}/_P = M & \longrightarrow & \overline{M} := \overline{\mathcal{G}}/_P = \partial_{CB}M\dot{\cup}M
\end{array}$$

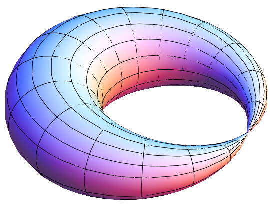
The boundary points are defined by inextendable curves in the bundle $\mathcal{G}$ which are of finite length. Please note, that two different inextendable curves of finite length may define the same boundary point. To define the equivalenz classes we can simply use the concept of Cauchy sequences as it is done for the Cauchy completion.

To fill the definition with life examples are given. The Cartan boundary of a homogeneous space is trivial ([Fra08]). Also in [Fra08] the Cartan boundary of the conformally flat space $\mathbb{R}^n$ of Riemannian signature was given. We generalize those arguments to describe the Cartan boundary of the conformally flat space $\mathbb{R}^{p,q}$ with arbitrary signature, identifying the Cartan boundary as a subset of the Möbius space $Q^{p,q}$ of signature (p,q):

$$\begin{aligned}\partial_{CB}\big(\mathbb{R}^{p,q},[\langle\cdot,\cdot\rangle_{p,q}]\big) \;=\;& Q^{p-1,q-1}\times S^1_+ \Big/ \sim \\ & \text{with } S^1_+ := \{e^{i\varphi} \mid 0\le\varphi\le\pi\} \\ & \text{and } (\mathbb{R}x, e^{i\varphi}) \sim (\mathbb{R}y, e^{i\psi}) \text{ if } \varphi,\psi\in\{0,\pi\},\end{aligned}$$

where $Q^{p-1,q-1}$ is the Möbius space of signature $(p-1,q-1)$. The conformal class $c_{p,q}$ is degenerated along S^1 and the restriction of $c_{p,q}$ to any subset $Q^{p-1,q-1}\times\{e^{i\varphi}\}$ for $\varphi\in(0,\pi)$ is exactly the conformal class $c_{p-1,q-1}$ of $Q^{p-1,q-1}$.

The Cartan boundary of $\mathbb{R}^{1,2}$ for example looks like this:



When discussing the Cartan boundary of space times as in [Sch73], [Cli66], [Cla78] just to name a few, physicists considered horizontal curves. We generalized this concept to Cartan geometries modeled on a reductive space, i.e. modeled on a homogeneous space $G/_P$ with $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{m}$ such that $\mathfrak{m}$ is $Ad(P)$-equivariant. In those Cartan geometries it is sufficient to consider the horizontal, inextendable curves of finite length when searching for boundary points. These ideas will be further specialized for semi-Riemannian manifolds where we will identify the geodesics as the projections of horizontal, ω-constant curves. Hence for (M,g) being a semi-Riemannian manifold all geodesics will be complete in $\overline{M}=M\dot\cup\partial_{CB}M$. In [Ger68] b.a. completeness was defined for Lorentzian manifolds, which is a stronger concept of completion than geodesical completeness. We find that for Lorentzian manifolds the Cartan completion $\overline{M}=M\dot\cup\partial_{CB}M$ is b.a. complete. As another example of a Cartan geometry modeled on a homogeneous space we discuss Riemannian manifolds and find that in this case the Cartan boundary is identical to the metrical boundary defined by the Riemannian metric.

As we just indicated the curves defining the boundary points can be further specialized, yielding several concepts of completeness.

Cartan completeness All curves of finite length can be extended.

ω-completeness All ω-constant curves of finite length can be extended.

horizontal completeness All horizontal curves of finite length can be extended.

geodesical completeness All horizontal, ω-constant curves of finite length can be extended.

For a manifold with a Cartan geometry modeled on a reductive space Cartan completeness and horizontal completeness are equivalent and imply ω-completeness.

$$\begin{array}{ccc} \text{Cartan completeness} & \Longleftrightarrow & \text{horizontal completeness} \\ \Downarrow\not\Uparrow & & \\ \omega - \text{completeness} & & \end{array}$$

For Lorentzian manifolds we have many different concepts of completeness, which are not equivalent.

$$\begin{array}{ccc} \text{Cartan completeness} & \Longleftrightarrow & \text{horizontal completeness} \\ \Downarrow\not\Uparrow & & \Downarrow\not\Uparrow \\ \text{b.a. completeness} & \not\Longleftrightarrow & \text{geodesical completeness} \\ \Downarrow\not\Uparrow & & \\ \text{timelike geodesical} & & \\ \text{completeness} & & \end{array}$$

In the Riemannian case all concepts coincide.

$$\begin{array}{ccc} \text{Cartan completeness} & \Longleftrightarrow & \text{horizontal completeness} \\ \Updownarrow & & \Updownarrow \\ \omega\text{-completeness} & \Longleftrightarrow & \text{geodesical completeness} \end{array}$$

In general there is a strict hierarchy.

$$\begin{array}{c} \text{Cartan completeness} \\ \Downarrow \ \not\Uparrow \\ \omega\text{-completeness} \\ \Downarrow \ \not\Uparrow \\ \text{geodesical completeness} \end{array}$$

The strictness of this hierarchy is verified by an example of [Cli66] describing a manifold with a trivial $\{id\}$-principal bundle as Cartan bundle, which is ω-complete but not Cartan complete and an example based on an idea of [Cli66] of a Cartan geometry modeled on a reductive space which is geodesically complete but not ω-complete.
A manifold is said to be locally complete at a point $x \in M$, if there is a neighborhood U of x such that the Cartan boundary of U is identical to the boundary defined by the embedding $U \hookrightarrow M$. A Cartan complete manifold is locally complete (at every point). And a closed subset of a locally complete manifold is also locally complete. However there are manifolds which are at no point locally complete. An example of [Sch73] of a manifold which is at no point locally complete is discussed.
For manifolds modeled on a reductive space the degeneration of boundary fibres is studied with the help of the singular holonomy group based on [Cla78] and [Cla79] where such considerations are done for space-times. We find that a boundary fibre is isomorphic to P divided by the singular holonomy group and it will degenerate at a curvature singularity.

An example of a space-time with degenerated boundary fibre is given by [Bos79] - a lorentzian manifold where future and past infinity are identified with each other in the Cartan boundary. We will discuss this manifold in detail. It is also an example of a manifold where the space $\overline{M} = M \dot{\cup} \partial_{CB} M$ is not be Hausdorff and not T_1.

Next we will study the Cartan boundary with the help of embeddings as it is done in [Fra08]. A Cartan geometry $(\mathcal{G}_M, \pi_M, M; \omega_M)$ is embedded into another Cartan geometry of the same type by a geometric embedding $\sigma : \mathcal{G}_M \longrightarrow \mathcal{G}_N$ which respects the Cartan connections and covers $s : M \longrightarrow N$. We can compare the Cartan boundary $\partial_{CB} M$ with the topological one obtained from the embedding, $\partial_{top} M \subset N$. In the topological boundary the accessible points are defined to be the endpoints of C^1-paths $\gamma : [0,1] \longrightarrow N$ with $\gamma([0,1)) \subset s(M)$. If the topological boundary is nonempty the accessible points form a dense subset of it. The regular points of the Cartan boundary are given by those Cauchy sequences $(u_n)_{n \in \mathbb{N}} \subset \mathcal{G}_M$ such that $\big(\sigma(u_n)\big)_{n \in \mathbb{N}} \subset \mathcal{G}_N$ converges. We find that M joined by the regular points is Hausdorff. We can prolong the embedding $s : M \longrightarrow N$ by the boundary map $\partial s : \{\text{regular points in } \partial_{CB} M\} \longrightarrow \partial_{top} M$ and obtain the following dense inclusions

$$\{\text{accessible points}\} \subset \partial s\big(\{\text{regular points}\}\big) \subset \partial_{top} M, \text{ given } \partial_{top} M \neq \emptyset.$$

Knowing the structure of the Cartan boundary can give informations on possible geometric embeddings. Given for example a strict conformal embedding of the flat space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle_n)$ into (N, g), we find that (N, g) has to be conformally equivalent to the round sphere.

Also in [Fra08] the Cartan boundary of ${}_{\Gamma}\backslash M$ for M being an open subset of a manifold L endowed with a Cartan geometry $(\mathcal{G}_L, \pi, L; \omega_L)$ and γ being a discrete subgroup of $Aut(L)$ preserving M and acting freely and properly on M is studied. Then we set $\mathcal{G}_M := \pi^{-1}(M)$ and $\big({}_{\Gamma}\backslash \mathcal{G}_M, \pi, {}_{\Gamma}\backslash M; \omega_L\big)$ is called the canonical Cartan geometry on ${}_{\Gamma}\backslash M$ induced by the Cartan geometry on L. If $\mathcal{G}_M \subset \mathcal{G}_L$ is dense we obtain $\overline{{}_{\Gamma}\backslash \mathcal{G}_M} = \overline{{}_{\Gamma}\backslash \mathcal{G}_L}$ and if further $\mathcal{G}_L$ is complete we have $\overline{{}_{\Gamma}\backslash \mathcal{G}_M} = {}_{\Gamma}\backslash \mathcal{G}_L$. Applying this to the conformal space ${}_{\Gamma}\backslash \mathbb{R}^n$ as done in [Fra08], i.e. $\Gamma \subset ISO(\mathbb{R}^n, \langle \cdot, \cdot \rangle_n)$ being a discrete subgroup and $Crit \subset \mathbb{R}^n$ being the set on which Γ is not acting freely we find that $\mathbb{R}^n \setminus Crit \subset \mathbb{R}^n$ is open and dense and

$$\partial_{CB}\big({}_{\Gamma}\backslash(\mathbb{R}^n \setminus Crit)\big) = Crit \dot{\cup} \{point\}.$$

If Γ is not compact $\overline{\big({}_{\Gamma}\backslash(\mathbb{R}^n \setminus Crit)\big)}$ will not be Hausdorff.

Another boundary physicists often use is the conformal boundary defined with the help of conformal embeddings. This boundary is not uniquely defined. However with the results from [Fra08] above two conformal boundaries of a manifold have to coincide on a dense subset.

In Chapter 7 we finally discuss the Cartan boundaries of CR manifolds and the corresponding Fefferman spaces. Now let $(M^{2n+1}, T_{10}, \theta)$ be a strictly pseudo-convex CR manifold and $\mathcal{F}$ the corresponding Fefferman space endowed with an S^1-action. The Cartan geometries of both are denoted by $(\mathcal{G}_M, \pi_M, M; \omega_M)$ and $(\mathcal{G}_{\mathcal{F}}, \pi_{\mathcal{F}}, \mathcal{F}; \omega_{\mathcal{F}})$ respectively. Further $\widetilde{i}$ denotes the fundamental vector field generated by $i \in i\mathbb{R} = LA(S^1)$. Then we have

- S^1 acts by conformal isomorphisms on $\mathcal{F}$.
- The norms of both Cartan connections are invariant under the action of S^1,

$$\mathcal{L}_{\widetilde{i}} \|\omega_M\| = 0 \text{ and } \mathcal{L}_{\widetilde{i}} \|\omega_{\mathcal{F}}\| = 0.$$

- The curvatures of both Cartan connections vanish if lifts of the fundamental vector field generated by $i \in LA(S^1)$ are inserted,

$$\Omega^{\omega_M}(\widetilde{i}, \cdot) = 0 \text{ and } \Omega^{\omega_{\mathcal{F}}}(\widetilde{i}, \cdot) = 0.$$

- S^1 acts by isometries on $\mathcal{G}_M$ and $\mathcal{G}_{\mathcal{F}}$ with respect to the Riemannian metrics induced by the Cartan connections.
- The S^1 actions on $\mathcal{G}_M$ and $\mathcal{G}_{\mathcal{F}}$ can be prolonged to the boundaries and the actions $S^1 \times \overline{\mathcal{G}_M} \longrightarrow \overline{\mathcal{G}_M}$ and $S^1 \times \overline{\mathcal{G}_{\mathcal{F}}} \longrightarrow \overline{\mathcal{G}_{\mathcal{F}}}$ are continuous.

Especially we prove the following proposition.

Proposition 1.2 *Let (M, H, θ) be a strictly pseudo-convex CR manifold joined with a complex line bundle $\mathcal{E}(1,0) \longrightarrow M$ together with a duality between $\mathcal{E}(1,0)^{\otimes(n+2)}$ and the canonical complex line bundle $\mathcal{K}$ of M. Then we have*

$$\overline{\mathcal{G}_M \times \{e\}} \subset \overline{\mathcal{G}_{\mathcal{F}}} \text{ and especially } \partial_{CB} M \subset pr(\partial_{CB}\mathcal{F}).$$

An example terminates the paper. The Heisenberg group is a flat CR manifold. So next to the homogeneous space this is one of the basic examples of CR manifolds. However the Heisenberg group is - contrary to the homogeneous space - not compact.
It can be realized as

$$He(n) := \left\{ \begin{pmatrix} 1 & X^t & z \\ 0 & I_n & Y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; \begin{matrix} z \in \mathbb{R} \\ X, Y \in \mathbb{R}^n \end{matrix} \right\} \subset Gl(n+2).$$

We find that the Cartan boundary of the Heisenberg group is a single point $\{\infty\}$ and the Cartan boundary of the corresponding Fefferman space is a nondegenerate S^1-fibre over ∞. Further $\overline{\mathcal{F}_{He(n)}} \longrightarrow \overline{He(n)}$ is a S^1-principal bundle and especially Hausdorff.

$$\begin{tikzcd} S^1 \circlearrowright \mathcal{F} \arrow[r] \arrow[d, "\pi"'] & \overline{\mathcal{F}} \simeq \mathcal{F}\dot{\cup} S^1 \circlearrowleft S^1 \arrow[d, "\overline{\pi}"] \\ He(n) \arrow[r] & \overline{He(n)} \simeq He(n)\dot{\cup}\{point\} \end{tikzcd}$$

Chapter 2

CR Manifolds and Fefferman Spaces

CR (*Cauchy-Riemann*) manifolds are an odd dimensional correspondent of Kähler manifolds. A Kähler manifold is a Riemannian manifold (M^n, g) endowed with an isometric and parallel almost complex structure J. Hence a Kähler manifold is complex and of even dimension. For manifolds with odd dimension the defining properties can be adjusted in order to obtain a CR manifold.
CR geometry combines knowledge of partial differential equations with higher dimensional complex analysis and differential geometry. At first mainly the aspects of partial differential equations and complex analysis were studied. In 1986 the discovery of CR submanifolds by A. Bejancu ([Bej86]) arouse a large interest and many investigations focusing on the differential geometry of CR manifolds followed. Amongst others a geometric relationship between CR geometry and conformal geometry via Fefferman spaces was discovered.
Fefferman spaces were first studied by Charles Fefferman ([Fef76]) while looking for CR invariants of strictly pseudo-convex hyperplanes in $\mathbb{C}^n$. Since then many authors enlarged his definition to more general geometric situations and the Fefferman spaces gained increased interest not solely in CR geometry (see for example [DT06]) but also within the conformal geometry. Their conformal holonomy is located in $SU(1, m)$. Thus they are the Lorentzian conformal analog of the Calabi Yau spaces (see for example [Arm07], [Bau99], [BL04], [Bau10], [Cap06], [Lei08]) which causes a particular interest.

2.1 CR Manifolds

In this section CR manifolds will be introduced and we will give an overview about familiar facts on them. Common examples are explained.

2.1.1 Definition and Examples

There is plenty of literature on CR manifolds. We will mainly stick to [DT06] and [Pet02]. Let us at first give two possible ways of defining a CR manifold and show the equivalence of both definitions.

Definition 2.1 *Let M^{2n+1} be a smooth real manifold of dimension $2n+1$. A complex CR structure on the manifold M is a complex subbundle $T_{10} \subset TM^{\mathbb{C}}$ satisfying the following conditions*

1. *The complex dimension of the subbundle T_{10} is n, $dim_{\mathbb{C}} T_{10} = n$.*
2. *The commutator of sections in T_{10} is again a section in this subbundle, $\big[\Gamma(T_{10}), \Gamma(T_{10})\big] \subset \Gamma(T_{10})$, i.e. T_{10} is involutive.*
3. *The complex subbundle T_{10} and its complex conjugate share solely the zero-section, $T_{10} \cap \overline{T_{10}} = \{0\}$, that is $T_{10} \oplus \overline{T_{10}}$ is a direct sum.*

Hence according to the definition of a complex CR structure $T_{10} \subset TM^{\mathbb{C}}$ on M^{2n+1} the subbundle $T_{10} \oplus \overline{T_{10}} \subset TM^{\mathbb{C}}$ is a complex subbundle of codimension one. We set

$$E^{\mathbb{C}} := TM^{\mathbb{C}} /_{T_{10} \oplus \overline{T_{10}}}.$$

Definition 2.2 *Let M^{2n+1} be a smooth real manifold of dimension $2n+1$. A real CR structure on M is a pair (H, J) satisfying*

1. *$H \subset TM$ is a real subbundle of codimension one.*

2. $J : H \longrightarrow H$ *is an almost complex bundle endomorphism,* $J^2 = -1$.
3. *The following integrability conditions hold:*
   (a) $[JX, Y] + [X, JY] \in \Gamma(H)$ *for all sections* $X, Y \in \Gamma(H)$
   (b) *The Nijenhuis tensor vanishes,*
   $$N_J(X,Y) := J\big([JX,Y]+[X,JY]\big) - [JX,JY] + [X,Y] = 0 \text{ for all } X,Y \in H.$$

We set $E := TM/_H$.
As can be seen in the next lemma, the definitions above are just different views of the same object.

Lemma 2.1 *Let M^{2n+1} be a real manifold. There is a bijective map between the complex and the real CR structures on M^{2n+1}.*

Proof: We will give an explicit description of the correspondence.

$$\begin{array}{ccrcl} \{\text{complex CR structures }\} & \longrightarrow & \{ \text{ real CR structures } \} & & \\ & & H & := & Re\left(T_{10} \oplus \overline{T_{10}}\right) \\ & & & = & \left\{U + \overline{U} \;\middle|\; U \in T_{10}\right\} \\ T_{10} & \mapsto & & & \\ & & J : H & \longrightarrow & H \\ & & J(U+\overline{U}) & := & i \cdot (U - \overline{U}) = iU + \overline{iU} \end{array}$$

Defined like this H is a real codimension 1 subbundle and J is an almost complex structure. So we only have to confirm the integrability conditions. Given two arbitrary vector fields $X, Y \in \Gamma(H)$, $X = U + \overline{U}$, $Y = V + \overline{V}$ we have

$$\begin{aligned} [JX,Y]+[X,JY] &= [iU - i\overline{U}, V + \overline{V}] + [U + \overline{U}, iV - i\overline{V}] \\ &= i\big([U,V] + [U,\overline{V}] - [\overline{U},V] - [\overline{U},\overline{V}] \\ &\qquad + [U,V] + [\overline{U},V] - [U,\overline{V}] - [\overline{U},\overline{V}]\big) \\ &= 2i\big([U,V] - [\overline{U},\overline{V}]\big) \\ &= 2J\big([U,V] + \overline{[U,V]}\big) \\ &\in \Gamma(H). \end{aligned}$$

I.e. the first integrability condition is satisfied. Applying this to the Nijenhuis tensor we obtain:

$$\begin{aligned} N_J(X,Y) &= J\big([JX,Y]+[X,JY]\big) - [JX,JY] + [X,Y] \\ &= J\big(2J\big([U,V] + \overline{[U,V]}\big)\big) - [iU - i\overline{U}, iV - i\overline{V}] + [U + \overline{U}, V + \overline{V}] \\ &= -2\big([U,V] + \overline{[U,V]}\big) + [U,V] - [U,\overline{V}] - [\overline{U},V] + [\overline{U},\overline{V}] \\ &\quad + [U,V] + [U,\overline{V}] + [\overline{U},V] + [\overline{U},\overline{V}] \\ &= -2\big([U,V] + \overline{[U,V]}\big) + 2[U,V] + 2[\overline{U},\overline{V}] \\ &= 0. \end{aligned}$$

Hence the integrability conditions are both satisfied and (H, J) is a real CR structure on the manifold M.
On the other hand we have:

$$\begin{array}{rcccl} & & \{\text{complex CR structures }\} & \longleftarrow & \{ \text{ real CR structures } \} \\ H^{\mathbb{C}} & \subset TM^{\mathbb{C}} & & & \\ J^{\mathbb{C}} : H^{\mathbb{C}} & \longrightarrow & H^{\mathbb{C}} & & \\ (J^{\mathbb{C}})^2 & = & -1 & \leftarrow & (H,J). \\ T_{10} & := & \left\{X \in H^{\mathbb{C}} \;\middle|\; J^{\mathbb{C}}(X) = iX\right\} & & \\ & = & \left\{X - iJX \;\middle|\; X \in H\right\} & & \end{array}$$

At first we need to prove that zero is the only vector shared by T_{10} and $\overline{T_{10}}$. Given $X \in T_{10}$ we have $J^{\mathbb{C}}(\overline{X}) = \overline{J^{\mathbb{C}}(X)} = \overline{iX} = -i\overline{X}$. In other words $\overline{X}$ is an eigenvector of $J^{\mathbb{C}}$ for the eigenvalue $-i$ and consequently not an element of T_{10}. Therefore, $T_{10} \cap \overline{T_{10}} = \{0\}$ is true and the complex dimension of T_{10} is n.
In the next step we are checking the integrability condition:
Let $X, Y \in \Gamma(T_{10})$ be two vector fields. Using the Nijenhuis tensor we obtain that the commutator of both vector fields $[X,Y]$ is an eigenvector of $J^{\mathbb{C}}$ for the eigenvalue i:

$$\begin{aligned}
J^{\mathbb{C}}\big([X,Y]\big) &= J^{\mathbb{C}}\Big(\underbrace{N_J^{\mathbb{C}}(X,Y)}_{=0} - J^{\mathbb{C}}\big([J^{\mathbb{C}}X,Y] + [X,J^{\mathbb{C}}Y]\big) + [J^{\mathbb{C}}X, J^{\mathbb{C}}Y] \Big) \\
&= \big([J^{\mathbb{C}}X,Y] + [X,J^{\mathbb{C}}Y]\big) + J^{\mathbb{C}}\big([J^{\mathbb{C}}X, J^{\mathbb{C}}Y]\big) \\
&= \big([iX,Y] + [X,iY]\big) + J^{\mathbb{C}}\big([iX,iY]\big) \\
&= 2i[X,Y] - J^{\mathbb{C}}\big([X,Y]\big)
\end{aligned}$$

So we have $J^{\mathbb{C}}\big([X,Y]\big) = i[X,Y]$ and consequently the commutator $[X,Y]$ is again a section in T_{10} for all $X, Y \in \Gamma(T_{10})$. Hence T_{10} is a complex CR structure on the manifold M.
Please note that applying one map after the other gives twice the identity.

□

We will now give some examples for CR manifolds, that is to say manifolds equipped with a CR structure.

Lemma 2.2 *Let (M^{2n+1}, g, ξ) be a Sasaki manifold. With $\phi := \nabla\xi : TM \longrightarrow TM$ we obtain a real CR structure (H, J) on M defined by $H := \xi^{\perp}$ and $J := \phi|_H$.*

Proof: Let (M^{2n+1}, g, ξ) be a Sasaki manifold, i.e. g is a Riemannian metric on the manifold M and $\xi \in \mathfrak{X}(M)$ is a Killing vector field of length one, that is $L_\xi g \equiv 0$ and $g(\xi,\xi) \equiv 1$. Furthermore for $\phi = \nabla\xi$ we require that $(\nabla_X\phi)(Y) = g(Y,\xi)X - g(X,Y)\xi$ $(*)$ holds.
With ξ being a Killing vector field ϕ is skew symmetric:

$$\begin{aligned}
0 &\equiv (L_\xi g)(X,Y) \\
&= g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) \\
&= g(\phi(X), Y) + g(X, \phi(Y)).
\end{aligned}$$

With ξ being a vector field of constant length we obtain that the image of ϕ is orthogonal to ξ, $0 = X\big(g(\xi,\xi)\big) = 2g(\phi(X),\xi)$. This leads to the vanishing of $\phi(\xi)$, since ϕ is skew symmetric and $0 = g(\phi(X),\xi) = -g(X,\phi(\xi))$ for all $X \in \mathfrak{X}(M)$.
Using the facts above we can compute ϕ^2.

$$\begin{aligned}
g(\phi^2(X),Y) &\overset{\phi \text{ skew sym.}}{=} -g\big(\phi(X),\phi(Y)\big) \\
&= -X\big(\underbrace{g(\phi(Y),\xi)}_{=0}\big) + g\big(\nabla_X(\phi(Y)),\xi\big) \\
&= g\big(\nabla_X(\phi(Y)),\xi\big) - \underbrace{g\big(\phi(\nabla_X Y),\xi\big)}_{=0} \\
&= g\big((\nabla_x\phi)(Y),\xi\big) \\
&\overset{(*)}{=} g\big(g(Y,\xi)X - g(X,Y)\xi, \xi\big) \\
&= g(Y,\xi)g(X,\xi) - g(X,Y) \\
&= g\big(g(X,\xi)\xi - X, Y\big)
\end{aligned}$$

With $Y \in \mathfrak{X}(M)$ being arbitrarily chosen we obtain

$$\phi^2(X) = -X + g(X,\xi)\xi \text{ for all } X \in \mathfrak{X}(M).$$

With $H := \xi^\perp$ and $J := \phi|_H$ the image of J is H and $J^2 = \big(\phi|_H\big)^2 = -id$, that is $J : H \longrightarrow H$ is an almost complex bundle endomorphism.
We now have to prove that $[JX, Y] + [X, JY]$ is a section in H for $X, Y \in \Gamma(H)$. Using $(*)$ we compute

$$\begin{aligned}
[JX,Y]+[X,JY] &= \nabla_{\phi X}Y - \nabla_Y(\phi X) + \nabla_X(\phi Y) - \nabla_{\phi Y}X \\
&= \nabla_{\phi X}Y - \nabla_{\phi Y}X - (\nabla_Y\phi)(X) \\
&\quad -\phi(\nabla_Y X) + (\nabla_X\phi)(Y) + \phi(\nabla_X Y) \\
&\overset{(*)}{=} \nabla_{\phi X}Y - \nabla_{\phi Y}X + \phi([X,Y]) \\
&\quad + \underbrace{g(Y,\xi)}_{=0}X - g(X,Y)\xi - \underbrace{g(X,\xi)}_{=0}Y + g(Y,X)\xi \\
&= \nabla_{\phi X}Y - \nabla_{\phi Y}X + \phi([X,Y])
\end{aligned}$$

With the image of ϕ being H it is sufficient to prove that $g(\nabla_{\phi X}Y - \nabla_{\phi Y}X, \xi)$ vanishes in order to see that $[JX, Y] + [X, JY]$ is an element of H.
This can be seen by considering $0 = (\nabla_X\xi)\big(\underbrace{g(\xi,Y)}_{\equiv 0}\big) = g\big(\nabla_{\nabla_X\xi}\xi, Y\big) + g\big(\xi, \nabla_{\nabla_X\xi}X\big)$.

We get

$$\begin{aligned}
g(\xi, \nabla_{\phi X}Y - \nabla_{\phi Y}X) &= -g\big(\nabla_{\nabla_X\xi}\xi, Y\big) + g\big(\nabla_{\nabla_Y\xi}\xi, X\big) \\
&= -g\big(J^2(X), Y\big) + g\big(J^2(Y), X\big) \\
&= g\big(X, Y\big) - g\big(Y, X\big) \\
&= 0
\end{aligned}$$

So $[JX, Y] + [X, JY]$ is orthogonal to ξ and therefore a section in H.
Let us now take a look at the Nijenhuis tensor. As we have seen above for $X, Y \in \Gamma(H)$ we have $[JX, Y] + [X, JY] = \nabla_{\phi X}Y - \nabla_{\phi Y}X + \phi([X, Y])$. Thus we have:

$$\begin{aligned}
N_J(X,Y) &= J\big([JX,Y]+[X,JY]\big) - [JX,JY] + [X,Y] \\
&= \phi\big(\nabla_{\phi X}Y\big) - \phi\big(\nabla_{\phi Y}X\big) + \phi^2([X,Y]) - [\phi X, \phi Y] + [X,Y]
\end{aligned}$$

We use the fact $\phi^2(X) = -X + g(X,\xi)\xi$.

$$\begin{aligned}
N_J(X,Y) &= \phi\big(\nabla_{\phi X}Y\big) - \phi\big(\nabla_{\phi Y}X\big) + g([X,Y],\xi)\xi - [\phi X, \phi Y] \\
&= \underline{\phi\big(\nabla_{\phi X}Y\big)} - \underbrace{\phi\big(\nabla_{\phi Y}X\big)} + g([X,Y],\xi)\xi \underline{-\nabla_{\phi X}(\phi Y)} + \underbrace{\nabla_{\phi Y}(\phi X)} \\
&= \underline{-\big(\nabla_{\phi X}\phi\big)(Y)} + \underbrace{\big(\nabla_{\phi Y}\phi\big)(X)} + g([X,Y],\xi)\xi
\end{aligned}$$

Applying $(*)$ leads to

$$\begin{aligned}
N_J(X,Y) &= -\underbrace{g(Y,\xi)}_{=0}\phi X + g(\phi X, Y)\xi + \underbrace{g(X,\xi)}_{=0}\phi Y - g(\phi Y, X)\xi + g([X,Y],\xi)\xi \\
&= \big(g(\phi X, Y) - g(\phi Y, X)\big)\xi + g([X,Y],\xi)\xi \\
&= \big(g(\nabla_X\xi, Y) - g(\nabla_Y\xi, X)\big)\xi + g([X,Y],\xi)\xi \\
&= g(\xi, -\nabla_X Y + \nabla_Y X)\xi + g([X,Y],\xi)\xi \\
&= 0
\end{aligned}$$

Hence the Nijenhuis tensor vanishes as wanted and we have given a CR manifold $(M^{2n+1}, H = \xi^\perp, J = \nabla\xi|_H)$.

$\square$

Lemma 2.3 *Let (N^{n+1}, J_N) be a complex manifold of complex dimension $n+1$ and $M \subset N$ a real hypersurface, especially M is a real manifold of dimension $dim_{\mathbb{R}} M = 2n + 1$. Then (H, J) with $H := TM \cap J_N(TM) \subset TM$ and $J := J_N|_H$ is a real CR structure on M.*

Proof: We are checking the integrability conditions. Let $X, Y \in \Gamma(H)$ be vector fields. I.e. $X, Y \in \Gamma(TM)$ and we have $X = J\widetilde{X}$, $Y = J\widetilde{Y}$ for some vector fields $\widetilde{X}, \widetilde{Y} \in \Gamma(TM)$. With N being a complex manifold we have on the one hand $[JX, Y] + [X, JY] = J(2[X, Y])$ with $2[X, Y] \in \Gamma(TM)$ and on the other hand $[JX, Y] + [X, JY] = [-\widetilde{X}, Y] + [X, -\widetilde{Y}] \in \Gamma(TM)$. Thus the first integrability condition $[JX, Y] + [X, JY] \in \Gamma(H) = \Gamma(TM \cap J_N TM)$ holds for all $X, Y \in \Gamma(H)$.
Since (N^{n+1}, J_N) is a complex manifold, the Nijenhuis tensor is vanishing on the whole manifold. Hence the second integrability condition is trivially satisfied.

$\square$

Another example of CR structures is constructed for Heisenberg groups. The Heisenberg groups can be realized in the following way

$$He(n) := \left\{ (X, Y, z) := \begin{pmatrix} 1 & X^t & z \\ 0 & I_n & Y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} z \in \mathbb{R} \\ X, Y \in \mathbb{R}^n \end{array} \right\} \subset Gl(n+2)$$

The corresponding Lie algebras are:

$$\mathfrak{he}(n) := LA(He(n)) = \left\{ \begin{pmatrix} 0 & X^t & z \\ 0 & 0 & Y \\ 0 & 0 & 0 \end{pmatrix} =: M(X, Y, z) \;\middle|\; \begin{array}{l} z \in \mathbb{R} \\ X, Y \in \mathbb{R}^n \end{array} \right\}$$

A basis of the Lie algebra $\mathfrak{he}(n)$ is given by the tuple

$$\big(X_i := M(e_i, 0, 0), Y_i := M(0, e_i, 0), Z := M(0, 0, 1) \;\big|\; i = 1, \ldots, n\big).$$

For the Lie brackets we have $[X_i, Y_i] = Z$ for $i = 1, \ldots, n$ and the remaining brackets vanish. The left invariant vector fields on $He(n)$ defined by X_i, Y_i and Z are denoted by $\widetilde{X_i}, \widetilde{Y_i}, \widetilde{Z} \in \mathfrak{X}(He(n))$ and are given by

$$\begin{aligned} \widetilde{X_i}(X, Y, z) &= dL_{(X,Y,z)} M(e_i, 0, 0) \\ &= \begin{pmatrix} 0 & e_i^t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= M(e_i, 0, 0) \\ &= \tfrac{d}{dt} R_{\exp(tM(e_i,0,0))}(X, Y, z)\big|_{t=0}, \\ \widetilde{Y_i}(X, Y, z) &= dL_{(X,Y,z)} M(0, e_i, 0) \\ &= M(0, e_i, x_i) \\ \text{and} \quad \widetilde{Z}(X, Y, z) &= dL_{(X,Y,z)} M(0, 0, 1) \\ &= M(0, 0, 1). \end{aligned}$$

We set $H := span\{\widetilde{X_i}, \widetilde{Y_i} \mid i = 1, \ldots, n\}$ and

$$\begin{array}{rcl} J : H & \longrightarrow & H \\ \widetilde{X_i} & \mapsto & \widetilde{Y_i} \\ \widetilde{Y_i} & \mapsto & -\widetilde{X_i} \end{array} .$$

Obviously H is a subbundle of codimension one and J an almost complex subbundle endomorphism. Furthermore by inserting the vector fields of the basis $\big(\widetilde{X_i}, \widetilde{Y_i} \;\big|\; i = 1, \ldots, n\big)$ we see, that the Nijenhuis tensor as well as the sum $[J\cdot, \cdot] + [\cdot, J\cdot]$ vanish. So all criteria for a real CR structure are satisfied.

Lemma 2.4 *With the data given above (H, J) is a real CR structure on the Heisenberg group $He(n)$.*

2.1.2 The Levi Form of a CR Manifold

In this subsection which is mainly based on [Bau02] we will define the Levi form of a CR manifold. This is a very useful tool since it yields a positive definite hermitian product on H if the CR manifold is strictly pseudo-convex. We will deal with the real and the complex case and translate one into the other.

Definition 2.3 *Let (M, T_{10}) be a CR manifold. The hermitian bilinear form*

$$\begin{array}{rcl} L : \Gamma(T_{10}) \times \Gamma(T_{10}) & \longrightarrow & \Gamma\left(TM^{\mathbb{C}}/_{T_{10}\oplus\overline{T_{10}}}\right) = \Gamma(E^{\mathbb{C}}) \\ L(X,Y) & := & i \cdot [X, \overline{Y}]_{E^{\mathbb{C}}}, \end{array}$$

where $X_{E^{\mathbb{C}}} := [X]$ denotes the class of equivalence in $E^{\mathbb{C}}$, is called the Levi form of (M, T_{10}).

The Levi form defined as above is hermitian since

$$L(Y,X) = i[Y, \overline{X}]_{E^{\mathbb{C}}} = -i[\overline{X}, Y]_{E^{\mathbb{C}}} = \overline{i[X, \overline{Y}]_{E^{\mathbb{C}}}} = \overline{L(X,Y)}.$$

There is another way of defining a Levi form for a CR manifold, if we have a pseudo-hermitian structure given on M.

Definition 2.4 *Let (M, H, J) be a CR manifold. A 1-form $\theta \in \Omega^1(M^{2n+1})$ is called a pseudo-hermitian form or pseudo-hermitian structure on (M, H, J), if*

1. *$\theta_x \neq 0$ for all $x \in M$*
2. *$\theta|_H = 0$*

Then (M, T_{10}, θ) is called a pseudo-hermitian CR manifold.

Lemma 2.5 *Let (M, H, J) be a CR manifold. A pseudo-hermitian form θ on M exists if and only if M is orientable. If θ and $\hat{\theta}$ are two pseudo-hermitian forms on M, there is a function $f \in C^\infty(M)$ with $\hat{\theta} = f \cdot \theta$.*

We can prolong a pseudo-hermitian form $\theta : \Gamma(TM) \longrightarrow C^\infty(M, \mathbb{R})$ to a complex linear form.

$$\begin{array}{rcl} \theta^{\mathbb{C}} : \Gamma(TM^{\mathbb{C}}) & \longrightarrow & C^\infty(M, \mathbb{C}) \\ X + iY & \mapsto & \theta(X) + i\theta(Y) \end{array}$$

Writing $T_{10} = \{X - iJX \mid X \in H\}$ we get immediately $\theta^{\mathbb{C}}(X - iJX) = \theta(\underbrace{X}_{\in H}) - i\theta(\underbrace{JX}_{\in H}) = 0$.
In other words $\theta^{\mathbb{C}}$ vanishes on T_{10}. In the same way it also vanishes on the conjugate space $\overline{T_{10}}$. However $\theta_x \neq 0$ for all $x \in M$. So the same holds for $\theta^{\mathbb{C}}$.
The other way round given a complex pseudo-hermitian form $\theta^{\mathbb{C}}$, i.e. a one-form $\theta^{\mathbb{C}}$ on the complexified tangent space which satisfies $\theta^{\mathbb{C}}|_{T_{10}\oplus\overline{T_{10}}} \equiv 0$ and $\theta^{\mathbb{C}}_x \neq 0$ for all $x \in M$, we define the real pseudo-hermitian form $\theta : \Gamma(TM) \longrightarrow C^\infty(M, \mathbb{R})$ via $\theta(X) := Re\big(\theta^{\mathbb{C}}(X)\big)$. Setting $H := \{U + \overline{U} \mid U \in T_{10}\}$ we get $\theta|_H \equiv 0$. We have to prove that for every $x \in M$ the one-form θ_x does not vanish completely. Choosing some $x \in M$, there exists a $X^{\mathbb{C}} \in T_x M^{\mathbb{C}}$ satisfying $\theta^{\mathbb{C}}(X^{\mathbb{C}}) \neq 0$. Assume that $\theta^{\mathbb{C}}(X^{\mathbb{C}})$ is purely imaginary. In this case $\theta^{\mathbb{C}}(iX^{\mathbb{C}})$ is real and nonzero and therefore $\theta(\underbrace{iX^{\mathbb{C}} + \overline{iX^{\mathbb{C}}}}_{\in T_x M}) \neq 0$. Consequently θ is nowhere vanishing, $\theta_x \neq 0$ for all $x \in M$.
From now on we will write θ instead of $\theta^{\mathbb{C}}$.

Definition 2.5 *Let (M, H, J, θ) be a pseudo-hermitian CR manifold. The Levi form of (M, H, J, θ) is defined via*

$$\begin{array}{rcl} L_\theta : \Gamma(T_{10}) \times \Gamma(T_{10}) & \longrightarrow & C^\infty(M, \mathbb{C}) \\ L_\theta(X, Y) & := & -i \cdot d\theta(X, \overline{Y}) \end{array}$$

By using the symbol θ as well for the map $E^{\mathbb{C}} \ni X_{E^{\mathbb{C}}} = [X] \mapsto \theta(X) \in \mathbb{C}$, we can write $\theta \circ L(X, Y) = L_\theta(X, Y)$, because for all $X, Y \in \Gamma(T_{10})$ it holds:

$$\begin{array}{rcl} L_\theta(X, Y) & = & -i \cdot d\theta(X, \overline{Y}) \\ & = & -i\big(X\big(\underbrace{\theta(\overline{Y})}_{=0}\big) - \overline{Y}\big(\underbrace{\theta(X)}_{=0}\big) - \theta\big([X, \overline{Y}]\big)\big) \\ & = & i\theta\big([X, \overline{Y}]\big) \\ & = & \theta \circ L(X, Y) \end{array}$$

The Levi form L_θ is hermitian as well since

$$L_\theta(X, Y) = i\theta[X, \overline{Y}] = i\theta\overline{[\overline{X}, Y]} = \overline{i\theta[\overline{X}, Y]} = \overline{-i\theta[Y, \overline{X}]} = \overline{i\theta[Y, \overline{X}]} = \overline{L_\theta(Y, X)}$$

for all sections $X, Y \in \Gamma(T_{10})$.
If we have another pseudo-hermitian structure $\hat{\theta} = f \cdot \theta$ with $f \in C^\infty(M)$ given on M, the Levi form transforms like that

$$L_{f \cdot \theta} = f \cdot L_\theta,$$

since $d\hat{\theta} = df \cdot \theta + f \cdot d\theta$ and θ is vanishing on T_{10}.

Definition 2.6 *A pseudo-hermitian CR manifold (M, H, J, θ) is called*

- *nondegenerate, if the Levi form L_θ is nondegenerate,*
- *strictly pseudo-convex, if the Levi form is positive definite, $L_\theta > 0$.*

If the CR manifold M in question is the smooth boundary of a domain $\Omega \subset \mathbb{C}^{n+1}$, then the manifold M with the CR structure for embedded CR manifolds as described above is strictly pseudo-convex if and only if $\Omega \subset \mathbb{C}^{n+1}$ is strictly pseudo-convex. ([Cap02]).
For a strictly pseudo-convex CR manifold the Levi form is a positiv definite hermitian product on T_{10}

$$\begin{array}{rcl} L_\theta : \Gamma(T_{10}) \times \Gamma(T_{10}) & \longrightarrow & C^\infty(M, \mathbb{C}) \\ X, Y & \mapsto & L_\theta(X, Y) \end{array}$$

In the real case we can define the Levi form of (M, H, J) by $G(X, Y) := -[X, JY]_E$. Given a pseudo-hermitian form θ we define the Levi form $G_\theta(X, Y) := d\theta(X, JY)$. As in the complex case we have $G_\theta = \theta \circ G$.

$$G_\theta(X, Y) = d\theta(X, JY) = X(\underbrace{\theta(JY)}_{=0}) - (JY)(\underbrace{\theta(X)}_{=0}) - \theta[X, JY] = -\theta[X, JY] = \theta \circ G(X, Y),$$

whereby X, Y are sections in H, $X, Y \in \Gamma(H)$. Furthermore the Levi form G_θ is totally real and symmetric, that is for all $X, Y \in \Gamma(H)$ we have $G_\theta(JX, JY) = G_\theta(X, Y) = G_\theta(Y, X)$, since on the one hand we can write

$$\begin{array}{rcl} G_\theta(JX, JY) & = & -\theta[JX, J^2Y] \\ & = & \theta[JX, Y] \\ & = & \theta[JX, Y] - \underbrace{\theta(\overbrace{[JX, Y] + [X, JY]}^{\in\Gamma(H)})}_{=0} \\ & = & -\theta[X, JY] \\ & = & G_\theta(X, Y). \end{array}$$

And on the other hand we can compute

$$\begin{aligned} G_\theta(JX, JY) &= -\theta[JX, J^2Y] \\ &= \theta[JX, Y] \\ &= -\theta[Y, JX] \\ &= G_\theta(Y, X). \end{aligned}$$

We can prolong the Levi form L_θ to sections in $T_{10} \oplus \overline{T_{10}}$ by

$$\begin{array}{rrcccl} & L_\theta(\overline{X}, \overline{Y}) & := & \overline{L_\theta(X, Y)} & = & L_\theta(Y, X) \\ \text{and} & L_\theta(\overline{X}, Y) & = & L_\theta(X, \overline{Y}) & = & 0 \end{array}$$

for all $X, Y \in \Gamma(T_{10})$.

Let us take a look at the relationship between the complex and the real Levi form. The sections in H can be written as $X + \overline{X}$, where X is a section in T_{10}. Given this we can write:

$$\begin{aligned} L_\theta(X + \overline{X}, Y + \overline{Y}) &= L_\theta(X, Y) + L_\theta(\overline{X}, \overline{Y}) \\ &= L_\theta(X, Y) + L_\theta(Y, X) \\ &= i\theta\big([X, \overline{Y}] + [Y, \overline{X}]\big) \\ &= i\theta\big([X - \overline{X}, Y + \overline{Y}] - \underbrace{[X, Y]}_{\in\Gamma(T_{10})} + \underbrace{[\overline{X}, \overline{Y}]}_{\in\Gamma(\overline{T_{10}})}\big) \\ &= i\theta\big([X - \overline{X}, Y + \overline{Y}]\big) \\ &= \theta\big([iX - i\overline{X}, Y + \overline{Y}]\big) \\ &\overset{\text{def } J}{=} \theta\big([J(X + \overline{X}), Y + \overline{Y}]\big) \\ &= -\theta\big([Y + \overline{Y}, J(X + \overline{X})]\big) \\ &= G_\theta(Y + \overline{Y}, X + \overline{X}) \\ &= G_\theta(X + \overline{X}, Y + \overline{Y}) \end{aligned}$$

Hence on $\Gamma(H) \times \Gamma(H)$ the Levi forms L_θ and G_θ coincide.

If the given CR manifold is furthermore strictly pseudo-convex, that is to say $L_\theta(X, X) > 0$ for all nonzero sections $X \in \Gamma(T_{10})$, $X \neq 0$, we also have $L_\theta(\overline{X}, \overline{X}) = L_\theta(X, X) > 0$. That means the Levi form is positiv definite on $\Gamma(T_{10} \oplus \overline{T_{10}}) \times \Gamma(T_{10} \oplus \overline{T_{10}})$ and every restriction as well. Therefore, the Levi form L_θ of a strictly pseudo-convex CR manifold is a positive definite hermitian product on $T_{10} \oplus \overline{T_{10}}$ and G_θ is a positive definite inner product on H.

2.1.3 The Reeb Vector Field of the Contact Form θ

Let $(M^{2n+1}, T_{10}, \theta)$ be a strictly pseudo-convex or a nondegenerate CR manifold. Then its differential $d\theta$ is nondegenerate on $\Gamma(H) \times \Gamma(H)$ since we have $d\theta(\cdot, \cdot) = -G_\theta(\cdot, J\cdot)$ and according to the definition the Levi form G_θ is positive definite or at least nondegenerate. Especially $\theta \wedge (d\theta)^n_x \not\equiv 0$ is true for all points $x \in M$ and therefore θ is a contact form on the strictly pseudo-convex CR manifold M. The characteristic vector field of this contact form is denoted by $T \in \mathfrak{X}(M)$ and uniquely defined by $\theta(T) \equiv 1$ and $d\theta(T, \cdot) \equiv 0$.

Definition 2.7 *The vector field T is called Reeb vector field of the contact form θ, or characteristic direction of the strictly pseudo-convex or nondegenerate CR manifold (M, T_{10}, θ).*

With the help of the Reeb vector field the tangent bundle can be written as a direct sum

- $TM = H \oplus \mathbb{R}T$,
- $TM^{\mathbb{C}} = T_{10} \oplus \overline{T_{10}} \oplus \mathbb{C}T$

and the real and the complex Levi form can be prolonged in the following way.

$$\begin{aligned} G_\theta : \Gamma(TM) \times \Gamma(TM) &\longrightarrow C^\infty(M, \mathbb{R}) \\ G_\theta(T, \cdot) &:= 0 \end{aligned}$$

$$\begin{aligned} L_\theta : \Gamma(TM^{\mathbb{C}}) \times \Gamma(TM^{\mathbb{C}}) &\longrightarrow C^\infty(M, \mathbb{C}) \\ L_\theta(T, \cdot) &:= 0 \end{aligned}$$

Lemma 2.6 *Let (M, H, J, θ) be a strictly pseudo-convex (or nondegenerate) CR manifold. For all sections $X, Y \in \Gamma(H)$ we have*

1. *The commutator of the Reeb vector field with any other vector field $X \in \Gamma(H)$ is a section in H, $[T, X] \in \Gamma(H)$.*
2. *The real and the complex Levi form coincide on H, $G_\theta(X, Y) = L_\theta(X, Y)$.*
3. *The real Levi form is totally real, $G_\theta(JX, JY) = G_\theta(X, Y)$.*
4. *The real Levi form satisfies $G_\theta(JX, Y) + G_\theta(X, JY) = 0$.*
5. *The following equation holds:*

$$G_\theta\big([T, X], Y\big) - G_\theta\big([T, Y], X\big) = G_\theta\big([T, JX], JY\big) - G_\theta\big([T, JY], JX\big).$$

Proof:

1. We want to prove that the commutator of the Reeb vector field with any other vector field has no component in $\mathbb{R}T$. According to the defining properties of the Reeb vector field T we have for any section $X \in \Gamma(H)$:

$$\begin{aligned} 0 &= d\theta(T, X) \\ &= T\big(\underbrace{\theta(X)}_{\equiv 0}\big) - X\big(\underbrace{\theta(T)}_{\equiv 1}\big) - \theta[T, X] \\ &= -\theta[T, X] \end{aligned}$$

   Therefore, the commutator $[T, X]$ has no component in $\mathbb{R}T$ and we have consequently $[T, X] \in \Gamma(H)$.

2. We have already proven that the real and the complex Levi form coincide.
3. That the real Levi form is totally real has been proven earlier.
4. The equality claimed, $G_\theta(JX, Y) + G_\theta(X, JY) = 0$, is a direct conclusion of the fact that the real Levi form is totally real.

$$\begin{aligned} G_\theta(JX, Y) + G_\theta(X, JY) &= G_\theta(J^2X, JY) + G_\theta(X, JY) \\ &= -G_\theta(X, JY) + G_\theta(X, JY) \\ &= 0 \end{aligned}$$

5. Let $X, Y \in \Gamma(H)$ be arbitrary sections. With the help of the Jacobi identity for vector fields we have:

$$\begin{aligned}
& G_\theta([T,X],Y) - G_\theta([T,Y],X) - G_\theta([T,JX],JY) + G_\theta([T,JY],JX) \\
= \; & -\theta\big[[T,X],JY\big] + \theta\big[[T,Y],JX\big] + \theta\big[[T,JX],J^2Y\big] - \theta\big[[T,JY],J^2X\big] \\
= \; & \theta\Big(- \big[[T,X],JY\big] + \big[[T,Y],JX\big] - \big[[T,JX],Y\big] + \big[[T,JY],X\big]\Big) \\
= \; & \theta\Big(\underbrace{\big[[T,Y],JX\big] + \big[[JX,T],Y\big]}_{\overset{\text{Jac.}}{=} - \big[[Y,JX],T\big]} + \underbrace{\big[[T,JY],X\big] + \big[[X,T],JY\big]}_{\overset{\text{Jac.}}{=} - \big[[JY,X],T\big]} \Big) \\
= \; & \theta\Big(\big[[JX,Y] + [X,JY],T\big]\Big)
\end{aligned}$$

According to the definition of a CR manifold the term $[JX,Y] + [X,JY]$ is a section in H. So with the first claim of this lemma and the pseudo-hermitian form θ vanishing on the subbundle H, the last line of our equation is zero and we obtain

$$G_\theta\big([T,X],Y\big) - G_\theta\big([T,Y],X\big) = G_\theta\big([T,JX],JY\big) - G_\theta\big([T,JY],JX\big).$$

□

The analogous statements are true for the complex Levi form L_θ.

Lemma 2.7 *Let (M, T_{10}, θ) be a strictly pseudo-convex (or nondegenerate) CR manifold. Let X, Y be sections in T_{10}. Then we have:*

1. *The commutator of the Reeb vector field with any other vector field $X \in \Gamma(T_{10})$ or $\overline{X} \in \Gamma(\overline{T_{10}})$ has no component in $\mathbb{C}T$, $[T,X], [T,\overline{X}] \in \Gamma(T_{10} \oplus \overline{T_{10}})$.*
2. *$L_\theta\big([T,X],Y\big) + L_\theta\big(X,[T,Y]\big) = T\big(L_\theta(X,Y)\big)$*
3. *$L_\theta\big([T,\overline{X}],Y\big) = L_\theta\big([T,\overline{Y}],X\big)$*
4. *$L_\theta\big([T,X],\overline{Y}\big) = L_\theta\big([T,Y],\overline{X}\big)$*

Proof:

1. The proof is analogous to the proof of the corresponding statement for G_θ.
2. We want to prove that $L_\theta\big([T,X],Y\big) + L_\theta\big(X,[T,Y]\big)$ is equal to $T\big(L_\theta(X,Y)\big)$. Let X, Y be arbitrary sections in T_{10}. Since $L_\theta(X,\overline{Y})$ is defined to be zero and the Levi form satisfies $L_\theta(X,Y) = i\theta([X,\overline{Y}])$ we can write

$$\begin{aligned}
L_\theta\left(\big[T,X\big],Y\right) + L_\theta\left(X,\big[T,Y\big]\right) &= L_\theta\left(\big[T,X\big]_{T_{10}},Y\right) + L_\theta\left(X,\big[T,Y\big]_{T_{10}}\right) \\
&= i\theta\left(\big[[T,X]_{T_{10}},\overline{Y}\big] + \big[X,\overline{[T,Y]_{T_{10}}}\big]\right).
\end{aligned}$$

The commutator of sections in T_{10} is again a section in T_{10} and analogously commutating sections in $\overline{T_{10}}$ yields a section in $\overline{T_{10}}$. Furthermore the pseudo-hermitian form θ vanishes on $\Gamma(T_{10})$ and $\Gamma(\overline{T_{10}})$. Hence we obtain

$$\begin{aligned}
L_\theta\left(\big[T,X\big],Y\right) + L_\theta\left(X,\big[T,Y\big]\right) &= i\theta\Big(\big[[T,X],\overline{Y}\big] + \big[X,\overline{[T,Y]}\big]\Big) \\
&= i\theta\Big(\big[[T,X],\overline{Y}\big] + \big[X,[\overline{T},\overline{Y}]\big]\Big).
\end{aligned}$$

For further calculations we need to know what the complex conjugation of the Reeb vector field T is. The complexified tangent bundle $TM^{\mathbb{C}}$ splits into the direct sum $TM^{\mathbb{C}} = T_{10} \oplus \overline{T_{10}} \oplus \mathbb{C}T = T_{10} \oplus \overline{T_{10}} \oplus \mathbb{C}\overline{T}$, i.e. there is a $\lambda \in \mathbb{C}$ with $\overline{T} = \lambda T$. This however already implies that λ is actually 1 and the Reeb vector field is invariant under complex conjugation since $\lambda = \theta(\lambda T) = \theta(\overline{T}) = \overline{\theta(T)} = 1$. Using furthermore the Jacobi identity we obtain:

$$\begin{aligned}
L_\theta\big([T,X],Y\big) + L_\theta\big(X,[T,Y]\big) &= i\theta\Big(\big[[T,X],\overline{Y}\big] + \big[X,[T,\overline{Y}]\big]\Big) \\
&= i\theta\Big(\big[\overline{Y},[X,T]\big] + \big[X,[T,\overline{Y}]\big]\Big) \\
&\overset{\text{Jac.}}{=} i\theta\big[T,[X,\overline{Y}]\big] \\
&= i\Big(-\underbrace{d\theta(T,[X,\overline{Y}])}_{\equiv 0} - [X,\overline{Y}]\big(\underbrace{\theta(T)}_{\equiv 1}\big) + T\big(\theta[X,\overline{Y}]\big)\Big) \\
&= iT\big(\theta[X,\overline{Y}]\big) \\
&= T\big(L_\theta(X,Y)\big).
\end{aligned}$$

This is the identity claimed.

3. Let $X, Y \in \Gamma(T_{10})$ be arbitrary sections. We are going to prove that the equation $L_\theta\big([T,\overline{X}],Y\big) = L_\theta\big([T,\overline{Y}],X\big)$ is true, using the same arguments as above.

$$\begin{aligned}
L_\theta\big([T,\overline{X}],Y\big) - L_\theta\big([T,\overline{Y}],X\big) &= L_\theta\big([T,\overline{X}]_{T_{10}},Y\big) - L_\theta\big([T,\overline{Y}]_{T_{10}},X\big) \\
&= i\theta\Big(\big[[T,\overline{X}]_{T_{10}},\overline{Y}\big] - \big[[T,\overline{Y}]_{T_{10}},\overline{X}\big]\Big) \\
&= i\theta\Big(\big[[T,\overline{X}],\overline{Y}\big] + \big[[\overline{Y},T],\overline{X}\big]\Big) \\
&\overset{\text{Jac.}}{=} i\theta\Big(-\big[[\overline{X},\overline{Y}],T\big]\Big) \\
&= i\theta\underbrace{\big[\overbrace{[\overline{Y},\overline{X}]}^{\in\Gamma(\overline{T_{10}})},T\big]}_{\in\Gamma(T_{10}\oplus\overline{T_{10}})} \\
&= 0
\end{aligned}$$

4. Finally we have to prove that $L_\theta\big([T,X],\overline{Y}\big) = L_\theta\big([T,Y],\overline{X}\big)$ holds for all sections $X, Y \in \Gamma(T_{10})$.

   For X, Y being arbitrary sections in the subbundle T_{10} we can write for the complex Levi form $L_\theta(\overline{X},\overline{Y}) = L_\theta(Y,X) = i\theta[Y,\overline{X}]$. So with this identity and the arguments applied for proving the two previous statements we obtain

$$\begin{aligned}
L_\theta\big([T,X],\overline{Y}\big) - L_\theta\big([T,Y],\overline{X}\big) &= L_\theta\left([T,X]_{\overline{T_{10}}},\overline{Y}\right) - L_\theta\left([T,Y]_{\overline{T_{10}}},\overline{X}\right) \\
&= i\theta\Big(\big[Y,[T,X]_{\overline{T_{10}}}\big] - \big[X,[T,Y]_{\overline{T_{10}}}\big]\Big) \\
&= i\theta\Big(\big[Y,[T,X]\big] - \big[X,[T,Y]\big]\Big) \\
&= i\theta\Big(\big[Y,[T,X]\big] + \big[X,[Y,T]\big]\Big) \\
&\overset{Jac}{=} i\theta\underbrace{\big[T,\overbrace{[Y,X]}^{\in\Gamma(T_{10})}\big]}_{\in\Gamma(T_{10}\oplus\overline{T_{10}})} \\
&= 0.
\end{aligned}$$

□

2.1.4 The Webster Metric and the Tanaka Webster Connection

In this subsection based on [Tan75] and [Web78] we will give the definition of the Webster metric and the Tanaka Webster connection of a strictly pseudo-convex CR manifold. We will furthermore discuss useful properties of both. Later on the Tanaka Webster connection will be used to define the Fefferman space of a strictly pseudo-convex CR manifold.
With the help of the Reeb vector field T and the Levi form L_θ we can define a metric for any strictly pseudo-convex CR manifold.

Definition 2.8 *Let (M, T_{10}, θ) be a strictly pseudo-convex (or nondegenerate) CR manifold. The nondegenerate hermitian form defined by*

$$g_\theta := L_\theta + \theta \otimes \overline{\theta} : \Gamma(TM^{\mathbb{C}}) \times \Gamma(TM^{\mathbb{C}}) \longrightarrow C^\infty(M, \mathbb{C})$$

is called Webster metric of (M, T_{10}, θ).

On the real tangent space TM we obtain the Webster metric as a restriction of g_θ. We use the same symbol g_θ for the restricted metric

$$g_\theta = L_\theta|_{\Gamma(TM)\times\Gamma(TM)} + \theta \otimes \theta = G_\theta + \theta \odot \theta.$$

Let the signature of the Levi form on T_{10} be (p, q). If we have actually given a strictly pseudo-convex CR manifold, the signature of the Levi form will be $(0, n)$. Due to $L_\theta(X, \overline{Y}) = 0$ and $L_\theta(\overline{X}, \overline{Y}) = L_\theta(Y, X)$ for all $X, Y \in \Gamma(T_{10})$, the complex signature of the Levi form L_θ is $(2p, 2q)$. Consequently using $\theta(T) = 1$ the signature of g_θ is $(2p, 2q + 1)$.
Choosing another pseudo-hermitian structure $\hat{\theta} = f \cdot \theta$ where $f \in C^\infty(M)$ is a smooth function on M, the Webster metric transforms as follows.

$$g_{\hat{\theta}} = L_{\hat{\theta}} + \hat{\theta} \otimes \overline{\hat{\theta}} = f \cdot L_\theta + f^2 \theta \otimes \overline{\theta}$$

This is a conformal transformation solely on the subbundle $T_{10} \oplus \overline{T_{10}}$.

Proposition 2.1 *Let (M, T_{10}, θ) be a nondegenerate pseudo-hermitian CR manifold and T the Reeb vector field of θ. Then there exists a uniquely determined covariant derivative $\nabla^W : \Gamma(T_{10}) \longrightarrow \Gamma\big((TM^{\mathbb{C}})^* \otimes T_{10}\big)$ satisfying the following conditions.*

1. *∇^W preserves the Levi form L_θ, that is for all sections $X, Y, Z \in \Gamma(T_{10})$ we have*

$$Z\big(L_\theta(X, Y)\big) = L_\theta\big(\nabla^W_Z X, Y\big) + L_\theta\big(X, \nabla^W_{\overline{Z}} Y\big).$$

2. *$\nabla^W_T X = pr_{T_{10}}[T, X]$ for all $X \in \Gamma(T_{10})$.*
3. *$\nabla^W_{\overline{Y}} X = pr_{T_{10}}[\overline{Y}, X]$ for all $X, Y \in \Gamma(T_{10})$.*

Here $pr_{T_{10}}$ denotes the projection onto T_{10}. Furthermore ∇^W is torsion free on T_{10},

$$[X, Y] = \nabla^W_X Y - \nabla^W_Y X \textit{ for all } X, Y \in \Gamma(T_{10}).$$

The covariant derivative ∇^W is called Tanaka Webster connection.

Proof:
At first we will proof the existence of the covariant derivative wanted. We define for all sections $X, Y \in \Gamma(T_{10})$

$$\begin{array}{rcl} \nabla^W_T X & := & pr_{T_{10}}[T, X], \\ \nabla^W_{\overline{Y}} X & := & pr_{T_{10}}[\overline{Y}, X], \\ \nabla^W_Y X \in \Gamma(T_{10}) & \text{given by} & L_\theta(\nabla^W_Y X, Z) = Y\big(L_\theta(X, Z)\big) - L_\theta\big(X, [\overline{Y}, Z]\big) \\ & & \text{for all sections } Z \in \Gamma(T_{10}). \end{array}$$

According to the definition $\nabla^W_{\cdot} X$ is $C^\infty(M,\mathbb{C})$-linear, $\nabla^W_A \cdot$ is additiv and we have $\nabla^W_A(\lambda X) = \lambda \nabla^W_A X + A(\lambda) X$. Thus it is a covariant derivative.
Further ∇^W satisfies all wanted conditions and is therefore unique as well.
We have to prove that the torsion of ∇^W vanishes on the subbundle T_{10}, that is to say the equation $[X,Y] = \nabla^W_X Y - \nabla^W_Y X$ holds for all $X, Y \in \Gamma(T_{10})$.
Since $[X,Y], \nabla^W_X Y$ and $\nabla^W_Y X$ are sections in T_{10} and the Levi form is nondegenerate on $\Gamma(T_{10}) \times \Gamma(T_{10})$ it is sufficient to show that $L_\theta([X,Y],Z) = L_\theta(\nabla^W_X Y - \nabla^W_Y X, Z)$ for all sections $Z \in \Gamma(T_{10})$. Using the definition of the Tanaka Webster connection this is equivalent to $L_\theta([X,Y],Z) = X\big(L_\theta(Y,Z)\big)\underline{-L_\theta\big(Y,[\overline{X},Z]\big)} - Y\big(L_\theta(X,Z)\big)\underline{+L_\theta\big(X,[\overline{Y},Z]\big)}$. We will take a closer look at the underlined parts.

$$
\begin{aligned}
L_\theta\left(X,[\overline{Y},Z]\right) - L_\theta\left(Y,[\overline{X},Z]\right) &= L_\theta\left(X, pr_{T_{10}}[\overline{Y},Z]\right) - L_\theta\left(Y, pr_{T_{10}}[\overline{X},Z]\right) \\
&= i\theta\left(\left[X, \overline{pr_{T_{10}}[\overline{Y},Z]}\right]\right) - i\theta\left(\left[Y, \overline{pr_{T_{10}}[\overline{X},Z]}\right]\right) \\
&= i\theta\Big(\left[X, pr_{\overline{T_{10}}}[Y,\overline{Z}]\right]\Big) - i\theta\Big(\left[Y, pr_{\overline{T_{10}}}[X,\overline{Z}]\right]\Big)
\end{aligned}
$$

We have $[Y,\overline{Z}] = pr_{T_{10}}[Y,\overline{Z}] + pr_{\overline{T_{10}}}[Y,\overline{Z}] + \underbrace{\theta([Y,\overline{Z}])}_{=g_\theta([Y,\overline{Z}],T)} \cdot T$.

Therefore, we obtain for the commutator:

$$
\begin{aligned}
\left[X, pr_{\overline{T_{10}}}[Y,\overline{Z}]\right] &= \left[X,[Y,\overline{Z}]\right] - \left[X, pr_{T_{10}}[Y,\overline{Z}]\right] - \left[X, \theta([Y,\overline{Z}])\cdot T\right] \\
&= \left[X,[Y,\overline{Z}]\right] - \underbrace{\left[X, pr_{T_{10}}[Y,\overline{Z}]\right]}_{\in\Gamma(T_{10})} - \theta([Y,\overline{Z}]) \underbrace{\left[X,T\right]}_{\in\Gamma(T_{10}\oplus\overline{T_{10}})} - X\big(\theta([Y,\overline{Z}])\big)\cdot T.
\end{aligned}
$$

Inserting this into the previous equation, keeping in mind that θ vanishes on $\Gamma(T_{10}) \oplus \Gamma(\overline{T_{10}})$, we get

$$
\begin{aligned}
L_\theta\left(X,[\overline{Y},Z]\right) - L_\theta\left(Y,[\overline{X},Z]\right) &= i\theta\Big(\left[X, pr_{\overline{T_{10}}}[Y,\overline{Z}]\right]\Big) - i\theta\Big(\left[Y, pr_{\overline{T_{10}}}[X,\overline{Z}]\right]\Big) \\
&= i\theta\Big(\left[X,[Y,\overline{Z}]\right] - X\big(\theta([Y,\overline{Z}])\big)\cdot T\Big) \\
&\quad -i\theta\Big(\left[Y,[X,\overline{Z}]\right] - Y\big(\theta([X,\overline{Z}])\big)\cdot T\Big) \\
&= i\theta\Big(\left[X,[Y,\overline{Z}]\right]\Big) - i\theta\Big(\left[Y,[X,\overline{Z}]\right]\Big) \\
&\quad -iX\big(\theta([Y,\overline{Z}])\big) + iY\big(\theta([X,\overline{Z}])\big) \\
&= i\theta\Big(\left[X,[Y,\overline{Z}]\right] - \left[Y,[X,\overline{Z}]\right]\Big) \\
&\quad -X\big(i\theta([Y,\overline{Z}])\big) + Y\big(i\theta([X,\overline{Z}])\big).
\end{aligned}
$$

The Jacobi identity and the fact $L_\theta(\cdot,*) = i\theta([\cdot,\overline{*}])$ yield

$$
\begin{aligned}
L_\theta\left(X,[\overline{Y},Z]\right) - L_\theta\left(Y,[\overline{X},Z]\right) &\overset{\text{Jac}}{=} i\theta\Big(\left[[X,Y],\overline{Z}\right]\Big) - X\big(L_\theta(Y,Z)\big) + Y\big(L_\theta(X,Z)\big) \\
&= L_\theta\big([X,Y],Z\big) - X\big(L_\theta(Y,Z)\big) + Y\big(L_\theta(X,Z)\big).
\end{aligned}
$$

Rearranging the terms gives the wanted equation.

$$
L_\theta\big([X,Y],Z\big) = X\big(L_\theta(Y,Z)\big) - L_\theta\left(Y,[\overline{X},Z]\right) - Y\big(L_\theta(X,Z)\big) + L_\theta\left(X,[\overline{Y},Z]\right)
$$

This is, as observed above, equivalent to the statement claimed. Hence for all $X, Y \in \Gamma(T_{10})$ the equation $[X,Y] = \nabla^W_X Y - \nabla^W_Y X$ holds, i.e. ∇^W is torsion free on T_{10}.

□

Via $\nabla^W_X T := 0$ and $\nabla^W_X \overline{U} := \overline{\nabla^W_{\overline{X}} U}$ for all vector fields $X \in \Gamma(TM^{\mathbb{C}})$ and all sections $U \in \Gamma(T_{10})$ we get a covariant derivative on $TM^{\mathbb{C}}$

$$
\nabla^W : \Gamma(TM^{\mathbb{C}}) \longrightarrow \Gamma\big((TM^{\mathbb{C}})^* \otimes TM^{\mathbb{C}}\big).
$$

Note that according to the definition of the Tanaka Webster connection the Reeb vector field T is parallel with respect to ∇^W and the image of ∇^W is actually in $\Gamma\big((TM^{\mathbb{C}})^* \otimes (T_{10} \oplus \overline{T_{10}})\big)$. The question arises wether we can restrict the Tanaka Webster connection to the subbundle H. Recall that sections in H can be expressed by sections in T_{10}. So for $X, Y \in \Gamma(H)$ we can find sections $U, V \in \Gamma(T_{10})$ such that $X = U + \overline{U}, Y = V + \overline{V}$. We can write

$$\begin{aligned} \nabla^W_X Y &= \nabla^W_{U+\overline{U}}(V + \overline{V}) \\ &= \nabla^W_U V + \nabla^W_U \overline{V} + \nabla^W_{\overline{U}} V + \nabla^W_{\overline{U}} \overline{V} \\ &= \nabla^W_U V + \overline{\nabla^W_{\overline{U}} V} + \nabla^W_{\overline{U}} V + \overline{\nabla^W_U V} \\ &= \underbrace{\nabla^W_U V}_{\in\Gamma(T_{10})} + \overline{\nabla^W_U V} + \underbrace{\nabla^W_{\overline{U}} V}_{\in\Gamma(T_{10})} + \overline{\nabla^W_{\overline{U}} V} \\ &\in \Gamma(H). \end{aligned}$$

Further we have

$$\begin{aligned} \nabla^W_T X &= \nabla^W_T U + \nabla^W_T \overline{U} \\ &= pr_{T_{10}}[T, U] + \overline{pr_{T_{10}}[T, U]} \\ &\in \Gamma(H). \end{aligned}$$

Thus the Tanaka Webster connection can also be seen as a connection on TM

$$\nabla^W : \Gamma(TM) \longrightarrow \Gamma\big(TM^* \otimes TM\big).$$

Here again the image of ∇^W is a subset of $\Gamma\big(TM^* \otimes H\big)$.

Lemma 2.8 *The Tanaka Webster connection $\nabla^W : \Gamma(TM^{\mathbb{C}}) \longrightarrow \Gamma\big((TM^{\mathbb{C}})^* \otimes TM^{\mathbb{C}}\big)$ is compatible with the Levi form L_θ, that is for all sections $X, Y, Z \in \Gamma(TM^{\mathbb{C}})$ we have $X\big(L_\theta(Y, Z)\big) = L_\theta(\nabla^W_X Y, Z) + L_\theta(Y, \nabla^W_{\overline{X}} Z)$. The restriction of the Tanaka Webster connection $\nabla^W : \Gamma(TM) \longrightarrow \Gamma\big(TM^* \otimes TM\big)$ is compatible with the real Levi form G_θ.*

Proof: In order to prove that the extended connection ∇^W is compatible with the Levi form L_θ we take arbitrary vector fields $X, Y, Z \in \Gamma(T_{10})$ and check the claim for these vector fields, the complex conjugated fields and the Reeb vector field.

1. According to the definition of the Tanaka Webster connection we have

$$X\big(L_\theta(Y, Z)\big) = L_\theta(\nabla^W_X Y, Z) + L_\theta(Y, \nabla^W_{\overline{X}} Z).$$

2. Recall that the Levi form and the Tanaka Webster connection were prolonged to sections in $\overline{T_{10}}$ by $L_\theta(\overline{Y}, \overline{Z}) = \overline{L_\theta(Y, Z)}$ and $\nabla^W_{\overline{X}} \overline{Y} = \overline{\nabla^W_X Y}$. With this we can write

$$\begin{aligned} \overline{X}\big(L_\theta(\overline{Y}, \overline{Z})\big) &= \overline{X\big(L_\theta(Y, Z)\big)} \\ &= \overline{L_\theta(\nabla^W_X Y, Z) + L_\theta(Y, \nabla^W_{\overline{X}} Z)} \\ &= L_\theta(\nabla^W_{\overline{X}} \overline{Y}, \overline{Z}) + L_\theta(\overline{Y}, \nabla^W_X \overline{Z}). \end{aligned}$$

3. We use the definition of the prolongation of the Levi form to sections in $\overline{T_{10}}$ again and further the fact that the Levi form is hermitian.

$$\begin{aligned} X\big(L_\theta(\overline{Y}, \overline{Z})\big) &= X\big(L_\theta(Z, Y)\big) \\ &= L_\theta(\nabla^W_X Z, Y) + L_\theta(Z, \nabla^W_{\overline{X}} Y) \\ &= L_\theta(\nabla^W_X \overline{Y}, \overline{Z}) + L_\theta(\overline{Y}, \nabla^W_{\overline{X}} \overline{Z}) \end{aligned}$$

4. Since the Levi form is hermitian this case is a direct conclusion of the second item.

$$\begin{aligned}
\overline{X}\big(L_\theta(Y,Z)\big) &= \overline{X\big(L_\theta(Z,Y)\big)}\\
&= L_\theta(\nabla^W_{\overline{X}}\overline{Z},\overline{Y}) + L_\theta(\overline{Z},\nabla^W_X\overline{Y})\\
&= L_\theta(\nabla^W_{\overline{X}}Y,Z) + L_\theta(Y,\nabla^W_X Z)
\end{aligned}$$

5. Let T be the Reeb vector field and $\lambda\in\mathbb{C}$. We use Lemma 2.7 and the fact that T_{10}, $\overline{T_{10}}$ and $\mathbb{C}T$ are orthogonal to each other with respect to the Levi form.

$$\begin{aligned}
\lambda T\big(L_\theta(X,Y)\big) &= \lambda L_\theta([T,X],Y) + \lambda L_\theta(X,[T,Y])\\
&= L_\theta(pr_{T_{10}}[\lambda T,X],Y) + L_\theta(X,pr_{T_{10}}[\overline{\lambda}T,Y])\\
&= L_\theta(\nabla^W_{\lambda T}X,Y) + L_\theta(X,\nabla^W_{\overline{\lambda}T}Y)
\end{aligned}$$

6. Since the Levi form is hermitian we can deduce the following equation directly from the item above

$$\begin{aligned}
\lambda T\big(L_\theta(\overline{X},\overline{Y})\big) &= \lambda T\big(L_\theta(Y,X)\big)\\
&= L_\theta(\nabla^W_{\lambda T}Y,X) + L_\theta(Y,\nabla^W_{\overline{\lambda}T}X)\\
&= L_\theta(\nabla^W_{\lambda T}\overline{X},\overline{Y}) + L_\theta(\overline{X},\nabla^W_{\overline{\lambda}T}\overline{Y}).
\end{aligned}$$

Using these informations and the facts that $L_\theta\left(\Gamma(T_{10}),\Gamma(\overline{T_{10}})\right)\equiv 0$ and $L_\theta(T,\cdot)\equiv 0$ we get for any sections $X = X_1+\overline{X_2}+xT, Y = Y_1+\overline{Y_2}+yT, Z = Z_1+\overline{Z_2}+zT\in\Gamma(TM^{\mathbb{C}})$ with $X_{1/2},Y_{1/2},Z_{1/2}\in\Gamma(T_{10})$ and $x,y,z\in\mathbb{C}$:

$$\begin{aligned}
X\big(L_\theta(Y,Z)\big) &= (X_1+\overline{X_2}+xT)\big(L_\theta(Y_1+\overline{Y_2}+yT,Z_1+\overline{Z_2}+zT)\big)\\
&= X_1\big(L_\theta(Y_1+\overline{Y_2},Z_1+\overline{Z_2})\big) + \overline{X_2}\big(L_\theta(Y_1+\overline{Y_2},Z_1+\overline{Z_2})\big)\\
&\quad +xT\big(L_\theta(Y_1+\overline{Y_2},Z_1+\overline{Z_2})\big)\\
&= X_1\big(L_\theta(Y_1,Z_1)\big) + X_1\big(L_\theta(\overline{Y_2},\overline{Z_2})\big)\\
&\quad +\overline{X_2}\big(L_\theta(Y_1,Z_1)\big) + \overline{X_2}\big(L_\theta(\overline{Y_2},\overline{Z_2})\big)\\
&\quad +xT\big(L_\theta(Y_1,Z_1)\big) + xT\big(L_\theta(\overline{Y_2},\overline{Z_2})\big).
\end{aligned}$$

With the statements above we can write:

$$\begin{aligned}
& X\big(L_\theta(Y,Z)\big)\\
= \;& L_\theta\left(\nabla^W_{X_1}Y_1,Z_1\right) + L_\theta\left(Y_1,\nabla^W_{\overline{X_1}}Z_1\right) + L_\theta\left(\nabla^W_{X_1}\overline{Y_2},\overline{Z_2}\right) + L_\theta\left(\overline{Y_2},\nabla^W_{\overline{X_1}}\overline{Z_2}\right)\\
& + L_\theta\left(\nabla^W_{\overline{X_2}}Y_1,Z_1\right) + L_\theta\left(Y_1,\nabla^W_{X_2}Z_1\right) + L_\theta\left(\nabla^W_{\overline{X_2}}\overline{Y_2},\overline{Z_2}\right) + L_\theta\left(\overline{Y_2},\nabla^W_{X_2}\overline{Z_2}\right)\\
& + L_\theta\left(\nabla^W_{xT}Y_1,Z_1\right) + L_\theta\left(Y_1,\nabla^W_{\overline{x}T}Z_1\right) + L_\theta\left(\nabla^W_{xT}\overline{Y_2},\overline{Z_2}\right) + L_\theta\left(\overline{Y_2},\nabla^W_{\overline{x}T}\overline{Z_2}\right)\\
= \;& L_\theta\left(\nabla^W_{X_1+\overline{X_2}}Y_1,Z_1\right) + L_\theta\left(Y_1,\nabla^W_{\overline{X_1+\overline{X_2}}}Z_1\right) + L_\theta\left(\nabla^W_{X_1+\overline{X_2}}\overline{Y_2},\overline{Z_2}\right) + L_\theta\left(\overline{Y_2},\nabla^W_{\overline{X_1+\overline{X_2}}}\overline{Z_2}\right)\\
& + L_\theta\left(\nabla^W_{xT}Y_1,Z_1\right) + L_\theta\left(Y_1,\nabla^W_{\overline{x}T}Z_1\right) + L_\theta\left(\nabla^W_{xT}\overline{Y_2},\overline{Z_2}\right) + L_\theta\left(\overline{Y_2},\nabla^W_{\overline{x}T}\overline{Z_2}\right).
\end{aligned}$$

Due to $\nabla^W_{\cdot}A\in\Gamma(T_{10})$ and $\nabla^W_{\cdot}\overline{A}\in\Gamma(\overline{T_{10}})$ for all $A\in\Gamma(T_{10})$ and using again $L_\theta\big(\Gamma(T_{10}),\Gamma(\overline{T_{10}})\big)\equiv 0$ we have

$$\begin{aligned}
X\big(L_\theta(Y,Z)\big) &= L_\theta\left(\nabla^W_{X_1+\overline{X_2}}Y_1,Z_1+\overline{Z_2}\right) + L_\theta\left(Y_1+\overline{Y_2},\nabla^W_{\overline{X_1+\overline{X_2}}}Z_1\right)\\
&\quad + L_\theta\left(\nabla^W_{X_1+\overline{X_2}}\overline{Y_2},Z_1+\overline{Z_2}\right) + L_\theta\left(Y_1+\overline{Y_2},\nabla^W_{\overline{X_1+\overline{X_2}}}\overline{Z_2}\right)\\
&\quad + L_\theta\left(\nabla^W_{xT}Y_1,Z_1+\overline{Z_2}\right) + L_\theta\left(Y_1+\overline{Y_2},\nabla^W_{\overline{x}T}Z_1\right)\\
&\quad + L_\theta\left(\nabla^W_{xT}\overline{Y_2},Z_1+\overline{Z_2}\right) + L_\theta\left(Y_1+\overline{Y_2},\nabla^W_{\overline{x}T}\overline{Z_2}\right)\\
&= L_\theta\left(\nabla^W_{X_1+\overline{X_2}+xT}(Y_1+\overline{Y_2}),Z_1+\overline{Z_2}\right)\\
&\quad + L_\theta\left(Y_1+\overline{Y_2},\nabla^W_{\overline{X_1+\overline{X_2}+xT}}(Z_1+\overline{Z_2})\right).
\end{aligned}$$

Since the Reeb vector field is parallel and $L_\theta(T,\cdot)\equiv 0$, the result wanted is obtained

$$\begin{aligned}
X\big(L_\theta(Y,Z)\big) &= L_\theta\left(\nabla^W_{X_1+\overline{X_2}+xT}(Y_1+\overline{Y_2}+yT), Z_1+\overline{Z_2}+zT\right)\\
&\quad +L_\theta\left(Y_1+\overline{Y_2}+yT, \nabla^W_{X_1+\overline{X_2}+xT}(Z_1+\overline{Z_2}+zT)\right)\\
&= L_\theta\left(\nabla^W_X Y, Z\right)+L_\theta\left(Y, \nabla^W_{\overline{X}} Z\right).
\end{aligned}$$

Hence the Tanaka Webster connection is compatible with the Levi form L_θ. Since G_θ coincides with L_θ on the real tangent space TM of M, the same holds for the real Levi form G_θ.

□

Definition 2.9 *The torsion of the Tanaka Webster connection is defined by*

$$Tor^W(U,V) := \nabla^W_U V - \nabla^W_V U - [U,V]$$

for all sections $U,V\in\Gamma(TM^{\mathbb{C}})$.

Proposition 2.1 states the vanishing of the torsion of the Tanaka Webster connection for sections $U,V\in\Gamma(T_{10})$, $Tor^W(U,V)=0$. Furthermore we can write for such sections:

$$\begin{aligned}
Tor^W(\overline{U},\overline{V}) &= \nabla^W_{\overline{U}}\overline{V}-\nabla^W_{\overline{V}}\overline{U}-[\overline{U},\overline{V}]\\
&= \overline{\nabla^W_U V}-\overline{\nabla^W_V U}-\overline{[U,V]}\\
&= \overline{Tor^W(U,V)}\\
&= 0
\end{aligned}$$

However the Tanaka Webster connection is not torsion free. We want to study the torsion of the Tanaka Webster connection in more detail. For this purpose we will take a closer look at the decomposition of sections in the complexified tangent space. According to the decomposition $TM^{\mathbb{C}}=T_{10}\oplus\overline{T_{10}}\oplus\mathbb{C}T$ any section $Z\in\Gamma(TM^{\mathbb{C}})$ can be written in the following way: $Z=pr_{T_{10}}(Z)+pr_{\overline{T_{10}}}(Z)+\theta(Z)T$. Especially projection and complex conjugation commute, that is to say $\overline{pr_{T_{10}}(Z)}=pr_{\overline{T_{10}}}(\overline{Z})$ for $Z\in\Gamma(TM^{\mathbb{C}})$, and the Reeb vector field is invariant under complex conjugation, $\overline{T}=T$. Now we can find further values for the torsion. Take again sections $U,V\in\Gamma(T_{10})$.

$$\begin{aligned}
Tor^W(U,\overline{V}) &= \nabla^W_U\overline{V}-\nabla^W_{\overline{V}}U-[U,\overline{V}]\\
&= \overline{\nabla^W_{\overline{U}}V}-pr_{T_{10}}[\overline{V},U]-[U,\overline{V}]\\
&= \overline{pr_{T_{10}}[\overline{U},V]}-pr_{T_{10}}[\overline{V},U]-[U,\overline{V}]\\
&= pr_{\overline{T_{10}}}[U,\overline{V}]+pr_{T_{10}}[U,\overline{V}]-[U,\overline{V}]\\
&= -\theta\big([U,\overline{V}]\big)T\\
&= iL_\theta(U,V)T\\
Tor^W(T,U) &= \nabla^W_T U-\underbrace{\nabla^W_U T}_{=0}-[T,U]\\
&= pr_{T_{10}}[T,U]-\underbrace{[T,U]}_{\in\Gamma(T_{10}\oplus\overline{T_{10}})}\\
&= -pr_{\overline{T_{10}}}[T,U]\\
Tor^W(T,\overline{U}) &= \nabla^W_T\overline{U}-\underbrace{\nabla^W_{\overline{U}} T}_{=0}-[T,\overline{U}]\\
&= \overline{\nabla^W_T U}-[T,\overline{U}]\\
&= \overline{pr_{T_{10}}[T,U]}-[T,\overline{U}]\\
&= pr_{\overline{T_{10}}}[T,\overline{U}]-\underbrace{[T,\overline{U}]}_{\in\Gamma(T_{10}\oplus\overline{T_{10}})}\\
&= -pr_{T_{10}}[T,\overline{U}]
\end{aligned}$$

To take a look at the real case, let $X = U + \overline{U}, Y = V + \overline{V} \in \Gamma(H)$ be sections in H with $U, V \in \Gamma(T_{10})$ convenient. It holds:

$$
\begin{aligned}
Tor^W(X,Y) &= \underbrace{Tor^W(U,V)}_{=0} + \underbrace{Tor^W(U,\overline{V})}_{=iL_\theta(U,V)T} + \underbrace{Tor^W(\overline{U},V)}_{=-i\overline{L_\theta(U,V)}T} + \underbrace{Tor^W(\overline{U},\overline{V})}_{=0} \\
&= iL_\theta(U,V)T - i\overline{L_\theta(U,V)}T \\
&= i\big(L_\theta(U,V) - L_\theta(\overline{U},\overline{V}) + \underbrace{L_\theta(U,\overline{V})}_{=0} - \underbrace{L_\theta(\overline{U},V)}_{=0}\big)T \\
&= L_\theta(iU - i\overline{U}, V + \overline{V})T \\
&= L_\theta(JX,Y)T \\
&= G_\theta(JX,Y)T.
\end{aligned}
$$

To give a formula for the torsion $Tor^W(T,X)$ with X a section in H, it is helpful to have an explicit way for calculating the projections. For any $Z \in \Gamma(T_{10} \oplus \overline{T_{10}}) = \Gamma(H^{\mathbb{C}})$, we have sections $\widetilde{U}, \widetilde{V} \in \Gamma(H)$ such that $Z = \widetilde{U} + i\widetilde{V}$. Moreover the sections $\widetilde{U}$ and $\widetilde{V}$ can be presented by sections U, V in T_{10} and therefore $Z = \widetilde{U} + i\widetilde{V} = U + \overline{U} + i(V + \overline{V})$. Hence we have

- $pr_{T_{10}}(Z) = U + iV$,
- $pr_{\overline{T_{10}}}(Z) = \overline{U} + i\overline{V}$ and
- $iJ^{\mathbb{C}}Z = i\big(iU - i\overline{U} + i(iV - i\overline{V})\big) = -U + \overline{U} - iV + i\overline{V}$.

Consequently we obtain $pr_{T_{10}}(Z) = \frac{1}{2}(Z - iJ^{\mathbb{C}}Z)$ and $pr_{\overline{T_{10}}}(Z) = \frac{1}{2}(Z + iJ^{\mathbb{C}}Z)$.
Now we can write for the torsion with $X \in \Gamma(H)$, $X = U + \overline{U}$

$$
\begin{aligned}
Tor^W(T,X) &= Tor^W(T,U) + Tor^W(T,\overline{U}) \\
&= -pr_{\overline{T_{10}}}[T,U] - pr_{T_{10}}[T,\overline{U}] \\
&= -\tfrac{1}{2}\big([T,U] + iJ^{\mathbb{C}}[T,U]\big) - \tfrac{1}{2}\big([T,\overline{U}] - iJ^{\mathbb{C}}[T,\overline{U}]\big) \\
&= -\tfrac{1}{2}\left([T,U] + [T,\overline{U}] + J^{\mathbb{C}}\big(i[T,U] - i[T,\overline{U}]\big)\right) \\
&= -\tfrac{1}{2}\big([T,U+\overline{U}] + J^{\mathbb{C}}([T,iU - i\overline{U}])\big) \\
&= -\tfrac{1}{2}\big([T,X] + J^{\mathbb{C}}(\underbrace{[T,JX]}_{\in\Gamma(H)}\big) \\
&= -\tfrac{1}{2}\big([T,X] + J[T,JX]\big).
\end{aligned}
$$

We present the obtained results in the following proposition.

Proposition 2.2 *Let (M, T_{10}, θ) be a nondegenerate pseudo-hermitian CR manifold and T the Reeb vector field of the pseudo-hermitian form θ. For all sections $Z \in \Gamma(T_{10} \oplus \overline{T_{10}})$ and all $W \in \Gamma(TM^{\mathbb{C}})$ we have*

- $pr_{T_{10}}(Z) = \frac{1}{2}(Z - iJ^{\mathbb{C}}Z)$,
- $pr_{\overline{T_{10}}}(Z) = \frac{1}{2}(Z + iJ^{\mathbb{C}}Z)$ *and*
- $\overline{pr_{T_{10}}(W)} = pr_{\overline{T_{10}}}(\overline{W})$.

For all sections $U, V \in \Gamma(T_{10})$ it holds

- $Tor^W(U,V) = 0$,
- $Tor^W(\overline{U},\overline{V}) = \overline{Tor^W(U,V)} = 0$,
- $Tor^W(U,\overline{V}) = iL_\theta(U,V)T$,

- $Tor^W(T,U) = -pr_{\overline{T_{10}}}[T,U]$ *and*
- $Tor^W(T,\overline{U}) = -pr_{T_{10}}[T,\overline{U}]$.

In addition we have for all sections $X,Y \in \Gamma(H)$

- $Tor^W(X,Y) = G_\theta(JX,Y)T$ *and*
- $Tor^W(T,X) = -\frac{1}{2}\big([T,X] + J[T,JX]\big)$.

We will continue by defining the curvature of the Tanaka Webster connection in the usual way.

Definition 2.10 *For all sections* $X,Y \in \Gamma(TM^{\mathbb{C}})$ *curvature operator of the Tanaka Webster connection is defined as*

$$\mathcal{R}^W(X,Y) := \nabla^W_X \nabla^W_Y - \nabla^W_Y \nabla^W_X - \nabla^W_{[X,Y]}.$$

Writing the Tanaka Webster curvature as a $(4,0)$*-tensor with respect to the metric* g_θ *we obtain*

$$\mathcal{R}^W(X,Y,Z,W) := g_\theta\big(\mathcal{R}^W(X,Y)Z,\overline{W}\big) \text{ for all sections } X,Y,Z,W \in \Gamma(TM^{\mathbb{C}}).$$

Since the Tanaka Webster connection has no component in the direction of the Reeb vector field, we can also write

$$\mathcal{R}^W(X,Y,Z,W) = L_\theta\big(\mathcal{R}^W(X,Y)Z,\overline{W}\big) \text{ for all sections } X,Y,Z,W \in \Gamma(TM^{\mathbb{C}}).$$

The following properties are met by the Tanaka Webster curvature.

Proposition 2.3 *For all sections* X,Y,Z,W *of* $TM^{\mathbb{C}}$ *it holds*

- $\mathcal{R}^W(X,Y,Z,W) = -\mathcal{R}^W(Y,X,Z,W) = -\mathcal{R}^W(X,Y,W,Z)$ *and*
- $\overline{\mathcal{R}^W(X,Y,Z,W)} = \mathcal{R}^W(\overline{X},\overline{Y},\overline{Z},\overline{W})$.

For all sections $A,B,C,D \in \Gamma(T_{10})$ *we have*

- $\mathcal{R}^W(A,\overline{B},C,\overline{D}) = \mathcal{R}^W(C,\overline{B},A,\overline{D})$ *and*
- $\mathcal{R}^W(A,B,\cdot,\cdot) \equiv 0$.

Proof:

- The first statement is true by definition since the endomorphism of the curvature is skew symmetric. We now prove that the Tanaka Webster curvature is skew symmetric in the third and forth component. Let X,Y,Z,W be arbitrary sections in the complex tangent space $TM^{\mathbb{C}}$ of M. Recall that the Tanaka Webster connection is compatible with the Levi form L_θ.

$$\begin{aligned}
\mathcal{R}^W(X,Y,Z,W) &= L_\theta\left(\nabla^W_X\nabla^W_Y Z,\overline{W}\right) - L_\theta\left(\nabla^W_Y\nabla^W_X Z,\overline{W}\right) - L_\theta\left(\nabla^W_{[X,Y]}Z,\overline{W}\right)\\
&= X\left(L_\theta\big(\nabla^W_Y Z,\overline{W}\big)\right) - L_\theta\big(\nabla^W_Y Z,\nabla^W_{\overline{X}}\overline{W}\big)\\
&\quad -Y\left(L_\theta\big(\nabla^W_X Z,\overline{W}\big)\right) + L_\theta\big(\nabla^W_X Z,\nabla^W_{\overline{Y}}\overline{W}\big)\\
&\quad -[X,Y]\left(L_\theta\big(Z,\overline{W}\big)\right) + L_\theta\big(Z,\nabla^W_{\overline{[X,Y]}}\overline{W}\big)\\
&= \underline{X\left(Y\big(L_\theta(Z,\overline{W})\big)\right)}\underbrace{-X\left(L_\theta\big(Z,\nabla^W_{\overline{Y}}\overline{W}\big)\right)}\\
&\quad \underbrace{-Y\left(L_\theta\big(Z,\nabla^W_{\overline{X}}\overline{W}\big)\right)} + L_\theta\big(Z,\nabla^W_{\overline{Y}}\nabla^W_{\overline{X}}\overline{W}\big)\\
&\quad \underline{-Y\left(X\big(L_\theta(Z,\overline{W})\big)\right)}\underbrace{+Y\left(L_\theta\big(Z,\nabla^W_{\overline{X}}\overline{W}\big)\right)}\\
&\quad \underbrace{+X\left(L_\theta\big(Z,\nabla^W_{\overline{Y}}\overline{W}\big)\right)} - L_\theta\big(Z,\nabla^W_{\overline{X}}\nabla^W_{\overline{Y}}\overline{W}\big)\\
&\quad \underline{-[X,Y]\left(L_\theta\big(Z,\overline{W}\big)\right)} + L_\theta\big(Z,\nabla^W_{\overline{[X,Y]}}\overline{W}\big)
\end{aligned}$$

The marked terms cancel out and we obtain

$$\begin{aligned}
\mathcal{R}^W(X,Y,Z,W) &= L_\theta\left(Z, \overline{\nabla^W_Y \nabla^W_X W - \nabla^W_X \nabla^W_Y W + \nabla^W_{[X,Y]} W}\right) \\
&= L_\theta\left(\nabla^W_Y \nabla^W_X W - \nabla^W_X \nabla^W_Y W + \nabla^W_{[X,Y]} W, \overline{Z}\right) \\
&= -L_\theta\left(\nabla^W_X \nabla^W_Y W - \nabla^W_Y \nabla^W_X W - \nabla^W_{[X,Y]} W, Z\right) \\
&= -\mathcal{R}^W(X,Y,W,Z).
\end{aligned}$$

- From the definition of the Tanaka Webster connection we get immediately, that for sections $X,Y \in \Gamma(TM^{\mathbb{C}})$ the following equation $\overline{\nabla^W_X Y} = \nabla^W_{\overline{X}} \overline{Y}$ holds. Hence we get directly the wanted result:

$$\begin{aligned}
\overline{\mathcal{R}^W(X,Y,Z,W)} &= \overline{L_\theta(\nabla^W_X \nabla^W_Y Z - \nabla^W_Y \nabla^W_X Z - \nabla^W_{[X,Y]} Z, \overline{W})} \\
&= L_\theta\left(\overline{\nabla^W_X \nabla^W_Y Z} - \overline{\nabla^W_Y \nabla^W_X Z} - \overline{\nabla^W_{[X,Y]} Z}, W\right) \\
&= L_\theta\left(\nabla^W_{\overline{X}} \nabla^W_{\overline{Y}} \overline{Z} - \nabla^W_{\overline{Y}} \nabla^W_{\overline{X}} \overline{Z} - \nabla^W_{[\overline{X},\overline{Y}]} \overline{Z}, W\right) \\
&= \mathcal{R}^W(\overline{X},\overline{Y},\overline{Z},\overline{W}).
\end{aligned}$$

- Let A,B,C,D be sections in T_{10}. We have to prove that the equation $\mathcal{R}^W(A,\overline{B},C,\overline{D}) = \mathcal{R}^W(C,\overline{B},A,\overline{D})$ holds.

$$\begin{aligned}
&\mathcal{R}^W(A,\overline{B},C,\overline{D}) - \mathcal{R}^W(C,\overline{B},A,\overline{D}) \\
&= L_\theta\left(\nabla^W_A \nabla^W_{\overline{B}} C - \nabla^W_{\overline{B}} \nabla^W_A C - \nabla^W_{[A,\overline{B}]} C, D\right) \\
&\quad -L_\theta\left(\nabla^W_C \nabla^W_{\overline{B}} A - \nabla^W_{\overline{B}} \nabla^W_C A - \nabla^W_{[C,\overline{B}]} A, D\right) \\
&= L_\theta\Big(\nabla^W_A \nabla^W_{\overline{B}} C - \nabla^W_C \nabla^W_{\overline{B}} A - \nabla^W_{\overline{B}} \overbrace{(\nabla^W_A C - \nabla^W_C A)}^{=[A,C]} \\
&\quad -\nabla^W_{[A,\overline{B}]} C + \nabla^W_{[C,\overline{B}]} A, D\Big)
\end{aligned}$$

We substitute $-\nabla^W_{\overline{B}}[A,C] = Tor^W([A,C],\overline{B}) - \nabla^W_{[A,C]}\overline{B} + [[A,C],\overline{B}]$ and obtain

$$\begin{aligned}
&\mathcal{R}^W(A,\overline{B},C,\overline{D}) - \mathcal{R}^W(C,\overline{B},A,\overline{D}) \\
&= L_\theta\Big(\nabla^W_A \nabla^W_{\overline{B}} C - \nabla^W_C \nabla^W_{\overline{B}} A + Tor^W([A,C],\overline{B}) - \nabla^W_{[A,C]}\overline{B} \\
&\quad +[[A,C],\overline{B}] - \nabla^W_{[A,\overline{B}]} C + \nabla^W_{[C,\overline{B}]} A, D\Big).
\end{aligned}$$

Since the torsion $Tor^W([A,C],\overline{B})$ is a multiple of the Reeb vector field T and $L_\theta(T,\cdot)$ vanishes, this term can be omitted. And with $\nabla^W_{[A,C]}\overline{B}$ being a section in $\overline{T_{10}}$ we have another term which is irrelevant and can be omitted. Furthermore we can use the Jacobi identity, $\big[[A,C],\overline{B}\big] = -\big[[C,\overline{B}],A\big] + \big[[A,\overline{B}],C\big]$. Hence we get

$$\begin{aligned}
&\mathcal{R}^W(A,\overline{B},C,\overline{D}) - \mathcal{R}^W(C,\overline{B},A,\overline{D}) \\
&= L_\theta\Big(\nabla^W_A \nabla^W_{\overline{B}} C - \nabla^W_C \nabla^W_{\overline{B}} A \underbrace{-\big[[C,\overline{B}],A\big] + \nabla^W_{[C,\overline{B}]} A}_{=Tor^W([C,\overline{B}],A)+\nabla^W_A[C,\overline{B}]} \\
&\quad + \underbrace{\big[[A,\overline{B}],C\big] - \nabla^W_{[A,\overline{B}]} C}_{=-Tor^W([A,\overline{B}],C)-\nabla^W_C[A,\overline{B}]}, D\Big).
\end{aligned}$$

Since the torsions in question have no component in T_{10}, they too can be omitted.

$$
\begin{aligned}
& \mathcal{R}^W(A,\overline{B},C,\overline{D}) - \mathcal{R}^W(C,\overline{B},A,\overline{D}) \\
= \ & L_\theta\Big(\nabla^W_A \nabla^W_{\overline{B}} C - \nabla^W_C \nabla^W_{\overline{B}} A + \nabla^W_A [C,\overline{B}] - \nabla^W_C [A,\overline{B}], D\Big) \\
= \ & L_\theta\left(\nabla^W_A \left(\nabla^W_{\overline{B}} C + [C,\overline{B}]\right) - \nabla^W_C \left(\nabla^W_{\overline{B}} A - [A,\overline{B}]\right), D\right) \\
= \ & L_\theta\Big(\nabla^W_A \big(\underbrace{Tor^W(\overline{B},C)}_{\text{multiple of } T} + \underbrace{\nabla^W_C \overline{B}}_{\in\Gamma(\overline{T_{10}})}\big) - \nabla^W_C \big(\underbrace{Tor^W(\overline{B},A)}_{\text{multiple of } T} + \underbrace{\nabla^W_A \overline{B}}_{\in\Gamma(\overline{T_{10}})}\big), D\Big) \\
= \ & L_\theta\Big(\underbrace{\nabla^W_A \nabla^W_C \overline{B} - \nabla^W_C \nabla^W_A \overline{B}}_{\in\Gamma(\overline{T_{10}})}, D\Big) \\
= \ & 0
\end{aligned}
$$

I.e. the equation $\mathcal{R}^W(A,\overline{B},C,\overline{D}) = \mathcal{R}^W(C,\overline{B},A,\overline{D})$ is true.

- Now we have to proof $\mathcal{R}^W(A,B,X,Y) = 0$ for all sections $A,B \in \Gamma(T_{10})$ and all $X,Y \in \Gamma(TM^{\mathbb{C}})$. Since A and B are restricted to $\Gamma(T_{10})$ we can use the defining properties of the Tanaka Webster connection and write

$$
\begin{aligned}
\mathcal{R}^W(A,B,X,Y) \ = \ & L_\theta\big(\nabla^W_A \nabla^W_B X, Y\big) - L_\theta\big(\nabla^W_B \nabla^W_A X, Y\big) - L_\theta\big(\nabla^W_{[A,B]} X, Y\big) \\
= \ & A\big(L_\theta(\nabla^W_B X, Y)\big) - L_\theta\big(\nabla^W_B X, [\overline{A}, Y]\big) \\
& -B\big(L_\theta(\nabla^W_A X, Y)\big) + L_\theta\big(\nabla^W_A X, [\overline{B}, Y]\big) \\
& -[A,B]\big(L_\theta(X,Y)\big) + L_\theta\big(X, \big[\overline{[A,B]}, Y\big]\big) \\
= \ & \underline{A\left(B\big(L_\theta(X,Y)\big)\right)} \underbrace{-A\left(L_\theta\big(X,[\overline{B},Y]\big)\right)}_{} \\
& \underbrace{-B\big(L_\theta(X,[\overline{A},Y])\big)}_{} + L_\theta\big(X, [\overline{B},[\overline{A},Y]]\big) \\
& \underline{-B\left(A\big(L_\theta(X,Y)\big)\right)} + \underbrace{B\left(L_\theta\big(X,[\overline{A},Y]\big)\right)}_{} \\
& + \underbrace{A\big(L_\theta(X,[\overline{B},Y])\big)}_{} - L_\theta\big(X, [\overline{A},[\overline{B},Y]]\big) \\
& \underline{-[A,B]\big(L_\theta(X,Y)\big)} + L_\theta\big(X, \big[\overline{[A,B]}, Y\big]\big).
\end{aligned}
$$

The marked terms cancel out and with the help of the Jacobi identity we get the result wanted.

$$
\begin{aligned}
\mathcal{R}^W(A,B,X,Y) \ & = \ L_\theta\big(X, [\overline{B},[\overline{A},Y]] - [\overline{A},[\overline{B},Y]] + [[\overline{A},\overline{B}],Y]\big) \\
& = \ L_\theta\big(X, [\overline{B},[\overline{A},Y]] + [\overline{A},[Y,\overline{B}]] + [Y,[\overline{B},\overline{A}]]\big) \\
& \overset{Jac}{=} \ 0
\end{aligned}
$$

Hence the statement $\mathcal{R}^W(A,B,X,Y) = 0$ is true for all sections $A,B \in \Gamma(T_{10})$ and all $X,Y \in \Gamma(TM^{\mathbb{C}})$.

□

To define the Ricci curvature and the scalar curvature as in the Riemannian case, we have to explain how traces are to be taken in the case of CR manifolds.
So let $\omega \in \Omega^2(TM^{\mathbb{C}},\mathbb{C})$ be a 2-form. Then there exists a uniquely defined endomorphism $\widetilde{\omega} \in End(\Gamma(T_{10}))$ with $\omega(U,\overline{V}) = L_\theta\big(\widetilde{\omega}(U), V\big)$ for all $U,V \in \Gamma(T_{10})$. Using this we define

the trace of ω with respect to θ via $Tr_\theta(\omega) := Tr(\widetilde{\omega})$. Choosing a unitary basis $(Z_1, \ldots, Z_n)$ of (T_{10}, L_θ) with $L_\theta(Z_i, Z_j) = \delta_{ij}\varepsilon_i$ we can write

$$Tr_\theta(\omega) = Tr(\widetilde{\omega}) = \sum_{i=1}^{n} \varepsilon_i L_\theta\left(\widetilde{\omega}(Z_i), Z_i\right) = \sum_{i=1}^{n} \varepsilon_i \omega(Z_i, \overline{Z_i}).$$

Definition 2.11 *The Webster Ricci curvature is defined as*

$$Ric^W := Tr_\theta^{(3,4)}(\mathcal{R}^W) = \sum_{i=1}^{n} \varepsilon_i \mathcal{R}^W(\cdot, \cdot, Z_i, \overline{Z_i}).$$

The scalar curvature of the Tanaka Webster connection, also called Webster scalar curvature, is defined by

$$R^W := Tr_\theta(Ric^W) = \sum_{i=1}^{n} \varepsilon_i Ric^W(Z_i, \overline{Z_i}).$$

Proposition 2.4 *The following properties hold.*

1. *The Webster Ricci curvature vanishes if both sections inserted are in T_{10} or both are in $\overline{T_{10}}$, $Ric^W(X, Y) = Ric^W(\overline{X}, \overline{Y}) = 0$ for all $X, Y \in \Gamma(T_{10})$.*
2. *We have $Ric^W(\overline{X}, \overline{Y}) = -\overline{Ric^W(X, Y)}$ for all $X, Y \in \Gamma(TM^{\mathbb{C}})$.*
3. *$Ric^W(U, V)$ is a map on M with pure imaginary values for all real vector fields of the CR manifold M, $U, V \in \mathfrak{X}(M)$.*
4. *For all sections $U, V \in \Gamma(H)$ the Webster Ricci curvature is invariant under the almost complex endomorphism J, $Ric^W(JU, JV) = Ric^W(U, V)$.*
5. *The Webster scalar curvature R^W is a real map on M.*

Proof:

1. With the properties of the Tanaka Webster curvature from Proposition 2.3 we have for all sections X, Y of the subbundle T_{10}

$$\begin{aligned} Ric^W(X, Y) &= \sum_{j=1}^{n} \varepsilon_j \underbrace{\mathcal{R}^W(X, Y, Z_j, \overline{Z_j})}_{=0,\text{ since } X,Y\in\Gamma(T_{10})} = 0 \\ \text{and} \quad Ric^W(\overline{X}, \overline{Y}) &= \sum_{j=1}^{n} \varepsilon_j \underbrace{\mathcal{R}^W(\overline{X}, \overline{Y}, Z_j, \overline{Z_j})}_{=\overline{\mathcal{R}^W(X,Y,\overline{Z_j},Z_j)}=0} = 0. \end{aligned}$$

2. For any vector fields X, Y in the complexified tangent space $TM^{\mathbb{C}}$ we can write

$$\begin{aligned} Ric^W(\overline{X}, \overline{Y}) &= \sum_{i=1}^{n} \varepsilon_i \mathcal{R}^W(\overline{X}, \overline{Y}, Z_i, \overline{Z_i}) \\ &= \overline{\sum_{i=1}^{n} \varepsilon_i \mathcal{R}^W(X, Y, \overline{Z_i}, Z_i)} \\ &= -\overline{\sum_{i=1}^{n} \varepsilon_i \mathcal{R}^W(X, Y, Z_i, \overline{Z_i})} \\ &= -\overline{Ric^W(X, Y)}. \end{aligned}$$

3. Let $U, V \in \mathfrak{X}(M)$ be real vector fields on M. Then we have sections $X, Y \in \Gamma(T_{10})$ and real valued maps $u, v \in C^\infty(M)$ such that $U = X + \overline{X} + uT$ and $V = Y + \overline{Y} + vT$.

We can write using the first two items of this proposition

$$\begin{aligned}
Ric^W(U,V) &= Ric^W\left(X+\overline{X}+uT, Y+\overline{Y}+vT\right)\\
&= \underbrace{Ric^W(X,Y)}_{=0} + Ric^W(X,\overline{Y}) + v\cdot Ric^W(X,T)\\
&\quad + Ric^W(\overline{X},Y) + \underbrace{Ric^W(\overline{X},\overline{Y})}_{=0} + v\cdot Ric^W(\overline{X},T)\\
&\quad + u\cdot Ric^W(T,Y) + u\cdot Ric^W(T,\overline{Y}) + uv\cdot \underbrace{Ric^W(T,T)}_{=0}\\
&= Ric^W(X,\overline{Y}) - \overline{Ric^W(X,\overline{Y})} + v\cdot\left(Ric^W(X,T) - \overline{Ric^W(X,T)}\right)\\
&\quad + u\cdot\left(Ric^W(T,Y) - \overline{Ric^W(T,Y)}\right)\\
&= 2i\cdot\Big(Im\big(Ric^W(X,\overline{Y})\big) + v\cdot Im\big(Ric^W(X,T)\big)\\
&\qquad + u\cdot Im\big(Ric^W(T,Y)\big)\Big)\\
&\in i\cdot\mathbb{R}.
\end{aligned}$$

4. We now want to prove that the Webster Ricci curvature is invariant under J. Let $U = X+\overline{X}$ and $V = Y+\overline{Y}$ be sections in H and $X, Y \in \Gamma(T_{10})$ suitable. Then we can write with the help of the first item of this proposition:

$$\begin{aligned}
Ric^W(JU,JV) &= Ric^W\left(iX - i\overline{X}, iY - i\overline{Y}\right)\\
&= -\underbrace{Ric^W(X,Y)}_{=0} + Ric^W(X,\overline{Y}) + Ric^W(\overline{X},Y) - \underbrace{Ric^W(\overline{X},\overline{Y})}_{=0}\\
&= Ric^W(X,\overline{Y}) + Ric^W(\overline{X},Y)\\
&= \underbrace{Ric^W(X,Y)}_{=0} + Ric^W(X,\overline{Y}) + Ric^W(\overline{X},Y) + \underbrace{Ric^W(\overline{X},\overline{Y})}_{=0}\\
&= Ric^W\left(X+\overline{X}, Y+\overline{Y}\right)\\
&= Ric^W(U,V).
\end{aligned}$$

5. According to the definition of the Webster scalar curvature and the properties of the Tanaka Webster curvature it holds:

$$\begin{aligned}
R^W &= \sum_{k,l=1}^{n} \varepsilon_k\varepsilon_l \mathcal{R}^W(Z_k,\overline{Z_k},Z_l,\overline{Z_l})\\
&= \sum_{k,l=1}^{n} \varepsilon_k\varepsilon_l \overline{\mathcal{R}^W(\overline{Z_k},Z_k,\overline{Z_l},Z_l)}\\
&= \sum_{k,l=1}^{n} \varepsilon_k\varepsilon_l \overline{\mathcal{R}^W(Z_k,\overline{Z_k},Z_l,\overline{Z_l})}\\
&= \overline{R^W}.
\end{aligned}$$

Hence the Webster scalar curvature has to be real.

□

2.1.5 The Transformation of the Tanaka Webster Connection

In this subsection based on [Lei05] we will check how the Reeb vector field, the Levi form and the Tanaka Webster connection behave under transformation of the pseudo-hermitian form θ.

Let $\hat{\theta} = e^{2f}\cdot\theta$ with $f \in \mathcal{C}(M)$ be another pseudo-hermitian form on the CR manifold M. For the Levi form we have $L_{\hat{\theta}} = e^{2f}\cdot L_\theta$ on $\Gamma(T_{10}\oplus\overline{T_{10}})\otimes\Gamma(T_{10}\oplus\overline{T_{10}})$ as we have seen already.

To study the behavior of the Tanaka Webster connection under change of the Levi form let $s = (Z_1, \ldots, Z_n)$ be a local section in (T_{10}, L_θ) which is unitary with respect to L_θ, then $\hat{s} = (\hat{Z}_1, \ldots, \hat{Z}_n)$ with $\hat{Z}_k := e^{-f} Z_k$ is a local unitary section in $(T_{10}, L_{\hat{\theta}})$.
The Reeb vector field belonging to $\hat{\theta}$ satisfies $\hat{\theta}(\hat{T}) \overset{!}{\equiv} 1$ according to the definition. I.e. there is a section $X \in \Gamma(T_{10} \oplus \overline{T_{10}})$ with $\hat{T} = e^{-2f} T + X$, where T is the Reeb vector field of θ. Furthermore $d\hat{\theta}(\hat{T}, \cdot) \overset{!}{\equiv} 0$ has to be true. Since $d\hat{\theta}$ is skew symmetric, it is sufficient to check this claim for sections $Y \in \Gamma(T_{10} \oplus \overline{T_{10}})$.

$$
\begin{aligned}
0 &\overset{!}{=} d\hat{\theta}(\hat{T}, Y) \\
&= \hat{T}\big(\underbrace{\hat{\theta}(Y)}_{\equiv 0}\big) - Y\big(\underbrace{\hat{\theta}(\hat{T})}_{\equiv 1}\big) - \hat{\theta}\big([\hat{T}, Y]\big) \\
&= -e^{2f}\theta\left(\left[e^{-2f}T + X, Y\right]\right) \\
&= -e^{2f}\theta\left(e^{-2f}[T, Y] - Y\left(e^{-2f}\right)T + [X, Y]\right) \\
&= \underbrace{-\theta\big([T, Y]\big)}_{= d\theta(T, Y) = 0} + e^{2f} Y\left(e^{-2f}\right) - e^{2f}\theta\big([X, Y]\big) \\
&= -2Y(f) - e^{2f}\theta\big(\underbrace{\nabla^W_X Y - \nabla^W_Y X}_{\in \Gamma(T_{10} \oplus \overline{T_{10}})} - Tor^W(X, Y)\big) \\
&= -2Y(f) + e^{2f}\theta\big(Tor^W(X, Y)\big)
\end{aligned}
$$

So it holds $2e^{-2f} Y(f) \overset{!}{=} \theta\big(Tor^W(X, Y)\big)$ for all $Y \in \Gamma(T_{10} \oplus \overline{T_{10}})$. Now we substitute for Y the sections Z_α and $\overline{Z_\alpha}$ of our unitary basis and write $X = \sum_\alpha \left(x_\alpha Z_\alpha + x_{\overline{\alpha}} \overline{Z_\alpha}\right)$. With the properties of the torsion from Proposition 2.2 we get with $Y = Z_\alpha$

$$
\begin{aligned}
2e^{-2f} Z_\alpha(f) &\overset{!}{=} \theta\big(Tor^W(X, Z_\alpha)\big) \\
&= \theta\big(Tor^W(pr_{\overline{T_{10}}} X, Z_\alpha)\big) \\
&= \theta\big(-iL_\theta(Z_\alpha, \overline{pr_{\overline{T_{10}}} X})T\big) \\
&= -iL_\theta(Z_\alpha, \overline{pr_{\overline{T_{10}}} X}) \\
&= -ix_{\overline{\alpha}},
\end{aligned}
$$

and with $Y = \overline{Z_\alpha}$ we obtain

$$
\begin{aligned}
2e^{-2f} \overline{Z_\alpha}(f) &\overset{!}{=} \theta\big(Tor^W(X, \overline{Z_\alpha})\big) \\
&= \theta\big(Tor^W(pr_{T_{10}} X, \overline{Z_\alpha})\big) \\
&= \theta\big(iL_\theta(pr_{T_{10}} X, Z_\alpha)T\big) \\
&= iL_\theta(pr_{T_{10}} X, Z_\alpha) \\
&= ix_\alpha.
\end{aligned}
$$

So we obtain $X = e^{-2f} \sum_\alpha \Big(-2i\overline{Z_\alpha}(f) Z_\alpha + 2iZ_\alpha(f)\overline{Z_\alpha}\Big)$ and hence

$$
\hat{T} = e^{-2f}\left(T + \sum_\alpha \big(-2i\overline{Z_\alpha}(f) Z_\alpha + 2iZ_\alpha(f)\overline{Z_\alpha}\big)\right).
$$

Now we determine the dual forms of $\hat{Z}_\alpha$ and $\overline{\hat{Z}_\alpha}$.
$\hat{\theta}^\alpha$ is defined by $\hat{\theta}^\alpha(\hat{Z}_\beta) \overset{!}{\equiv} \delta_{\alpha\beta}$, $\hat{\theta}^\alpha\left(\overline{\hat{Z}_\beta}\right) \overset{!}{\equiv} 0$ and $\hat{\theta}^\alpha(\hat{T}) \overset{!}{\equiv} 0$. I.e. we can write $\hat{\theta}^\alpha = e^f \theta^\alpha + x\theta$ for a map $x \in \mathcal{C}(M)$, which is determined by

$$
\begin{aligned}
0 &\overset{!}{=} \hat{\theta}^\alpha(\hat{T}) \\
&= \hat{\theta}^\alpha\left(e^{-2f}\left(T + \sum_\beta \left(-2i\overline{Z_\beta}(f) Z_\beta + 2iZ_\beta(f)\overline{Z_\beta}\right)\right)\right) \\
&= e^{-2f}\left(x - 2ie^f\,\overline{Z_\alpha}(f)\right).
\end{aligned}
$$

This results in $\hat{\theta}^\alpha = e^f \left(\theta^\alpha + 2i\overline{Z_\alpha}(f)\theta\right)$.

Analogously for $\hat{\theta}^{\overline{\alpha}}$ the equations $\hat{\theta}^{\overline{\alpha}}\left(\hat{\overline{Z_\beta}}\right) \overset{!}{\equiv} \delta_{\alpha\beta}$, $\hat{\theta}^{\overline{\alpha}}(\hat{Z}_\beta) \overset{!}{\equiv} 0$ and $\hat{\theta}^\alpha(\hat{T}) \overset{!}{\equiv} 0$ have to be true. We write consequently $\hat{\theta}^{\overline{\alpha}} = e^f\theta^{\overline{\alpha}} + x\theta$ for another map $x \in \mathcal{C}(M)$ and find $x = -2ie^f Z_\alpha(f)$ due to

$$\begin{aligned} 0 &\overset{!}{=} \hat{\theta}^{\overline{\alpha}}(\hat{T}) \\ &= \hat{\theta}^{\overline{\alpha}}\left(e^{-2f}\left(T + \sum_\beta \left(-2i\overline{Z_\beta}(f)Z_\beta + 2iZ_\beta(f)\overline{Z_\beta}\right)\right)\right) \\ &= e^{-2f}\left(x + 2ie^f df(Z_\alpha)\right). \end{aligned}$$

We get $\hat{\theta}^{\overline{\alpha}} = e^f\left(\theta^{\overline{\alpha}} - 2iZ_\alpha(f)\theta\right)$.

With those results we can describe the Levi form on the hole complexified tangent space.

$$\begin{aligned} L_{\hat{\theta}} &= \sum_\alpha \left(\hat{\theta}^\alpha \otimes \overline{\hat{\theta}^\alpha} + \hat{\theta}^{\overline{\alpha}} \otimes \overline{\hat{\theta}^{\overline{\alpha}}}\right) \\ &= e^{2f}\sum_\alpha \Big(\theta^\alpha \otimes \overline{\theta^\alpha} - 2iZ_\alpha(f)\theta^\alpha \otimes \overline{\theta} + 2i\overline{Z_\alpha}(f)\theta \otimes \overline{\theta^\alpha} + 4Z_\alpha(f)\overline{Z_\alpha}(f)\theta\otimes\overline{\theta} \\ &\qquad + \theta^{\overline{\alpha}} \otimes \overline{\theta^{\overline{\alpha}}} + 2i\overline{Z_\alpha}(f)\theta^{\overline{\alpha}} \otimes \overline{\theta} - 2iZ_\alpha(f)\theta \otimes \overline{\theta^{\overline{\alpha}}} + 4Z_\alpha(f)\overline{Z_\alpha}(f)\theta\otimes\overline{\theta} \Big) \\ &= e^{2f}L_\theta + e^{2f}\sum_\alpha \Big(2i\left(\overline{Z_\alpha}(f)\theta^{\overline{\alpha}} - Z_\alpha(f)\theta^\alpha\right)\otimes\overline{\theta} \\ &\qquad + 2i\theta \otimes \left(\overline{Z_\alpha}(f)\overline{\theta^\alpha} - Z_\alpha(f)\overline{\theta^{\overline{\alpha}}}\right) \\ &\qquad + 8Z_\alpha(f)\overline{Z_\alpha}(f)\theta\otimes\overline{\theta} \Big) \end{aligned}$$

The Tanaka Webster connection $\hat{\nabla}^W$ was defined for sections $X, Y \in \Gamma(T_{10})$ via

$$\hat{\nabla}^W_Y X \in \Gamma(T_{10}) \quad \text{given by} \quad \begin{array}{l} L_{\hat{\theta}}(\hat{\nabla}^W_Y, X, Z) = Y\big(L_{\hat{\theta}}(X,Z)\big) - L_{\hat{\theta}}(X, [\overline{Y}, Z]) \\ \text{for any } Z \in \Gamma(T_{10}). \end{array}$$

We can write

$$\begin{aligned} L_\theta\left(\hat{\nabla}^W_Y X, Z\right) &= e^{-2f}L_{\hat{\theta}}\left(\hat{\nabla}^W_Y X, Z\right) \\ &= e^{-2f}\left(Y\big(L_{\hat{\theta}}(X,Z)\big) - L_{\hat{\theta}}(X, [\overline{Y}, Z])\right) \\ &= e^{-2f}\Big(Y\big(e^{2f}L_\theta(X,Z)\big) - e^{2f}L_\theta(X, [\overline{Y}, Z]) \\ &\qquad + e^{2f}\sum_\alpha 2iZ_\alpha(f)\theta^\alpha(X)\cdot\overline{\theta}\left([\overline{Y}, Z]\right)\Big) \\ &= 2Y(f)L_\theta(X,Z) + \underbrace{Y\big(L_\theta(X,Z)\big) - L_\theta(X, [\overline{Y}, Z])}_{=L_\theta(\nabla^W_Y X, Z)} \\ &\qquad + 2i\underbrace{\left(\sum_\alpha \theta^\alpha(X)Z_\alpha\right)}_{=X}(f)\cdot\overline{\theta}\left([\overline{Y}, Z]\right) \\ &= 2Y(f)L_\theta(X,Z) + L_\theta(\nabla^W_Y X, Z) + 2iX(f)\cdot\overline{\theta}\left([\overline{Y}, Z]\right). \end{aligned}$$

The commutator $[\overline{Y}, Z]$ can be replaced using the torsion,

$$[\overline{Y}, Z] = \underbrace{\nabla^W_{\overline{Y}} Z - \nabla^W_Z \overline{Y}}_{\in\Gamma(T_{10}\oplus\overline{T_{10}})} - Tor^W(\overline{Y}, Z),$$

and the derivatives can be omitted since they have no component in $\mathbb{C}T$.

$$
\begin{aligned}
L_\theta\left(\hat\nabla^W_Y X, Z\right) &= 2Y(f)L_\theta(X,Z) + L_\theta(\nabla^W_Y X, Z) \\
&\quad -2iX(f)\cdot\overline{\theta}\big(\underbrace{Tor^W(\overline{Y},Z)}_{=-iL_\theta(Z,Y)T}\big) \\
&= 2Y(f)L_\theta(X,Z) + L_\theta(\nabla^W_Y X, Z) + 2iX(f)\cdot\overline{\theta}\left(iL_\theta(Z,Y)T\right) \\
&= 2Y(f)L_\theta(X,Z) + L_\theta(\nabla^W_Y X, Z) + 2X(f)\overline{L_\theta(Z,Y)} \\
&= 2Y(f)L_\theta(X,Z) + L_\theta(\nabla^W_Y X, Z) + 2X(f)L_\theta(Y,Z) \\
&= L_\theta\big(2Y(f)X + 2X(f)Y + \nabla^W_Y X, Z\big)
\end{aligned}
$$

Hence we obtain for sections $X, Y \in \Gamma(T_{10})$

$$\hat\nabla^W_Y X = \nabla^W_Y X + 2Y(f)X + 2X(f)Y.$$

In order to make the following calculations more transparent, we introduce the denotation: $\delta f := \sum_\alpha \overline{Z_\alpha}(f) Z_\alpha \in \Gamma(T_{10})$. Using this we ca write for example

$$\hat T = e^{-2f}(T - 2i\delta f + 2i\overline{\delta f}).$$

Now we check further how the Tanaka Webster Connection transforms. With $X, Y \in \Gamma(T_{10})$ we have for $\hat\nabla^W_{\overline{Y}} X$:

$$
\begin{aligned}
\hat\nabla^W_{\overline{Y}} X &:= \hat{pr}_{T_{10}}[\overline{Y}, X] \\
&= \hat{pr}_{T_{10}}\left(\nabla^W_{\overline{Y}} X - \nabla^W_X \overline{Y} + Tor^W(X, \overline{Y})\right) \\
&= \hat{pr}_{T_{10}}\left(\nabla^W_{\overline{Y}} X - \nabla^W_X \overline{Y} + iL_\theta(X,Y)T\right) \\
&= \hat{pr}_{T_{10}}\left(\nabla^W_{\overline{Y}} X - \nabla^W_X \overline{Y} + iL_\theta(X,Y)\left(e^{2f}\hat T + 2i\delta f - 2i\overline{\delta f}\right)\right) \\
&= \nabla^W_{\overline{Y}} X - 2L_\theta(X,Y)\delta f.
\end{aligned}
$$

For the calculations of $\hat\nabla^W_{\hat T} X$ recall that according to Proposition 2.2 the commutator $[\hat T, X]$ with X a section in T_{10} is an element of $\Gamma(T_{10} \oplus \overline{T_{10}})$. Hence it holds for the projections $\hat{pr}_{T_{10}}[\hat T, X] = pr_{T_{10}}[\hat T, X]$. Consequently we can write:

$$
\begin{aligned}
\hat\nabla^W_{\hat T} X &=: \hat{pr}_{T_{10}}[\hat T, X] \\
&= pr_{T_{10}}[\hat T, X] \\
&= pr_{T_{10}}\left(\left[e^{-2f}\left(T - 2i\delta f + 2i\overline{\delta f}\right), X\right]\right) \\
&= pr_{T_{10}}\left(e^{-2f}[T,X] - X(e^{-2f})T - 2ie^{-2f}[\delta f, X] + X(2ie^{-2f})\delta f\right. \\
&\qquad \left. +2ie^{-2f}[\overline{\delta f}, X] - X(2ie^{-2f})\overline{\delta f}\right) \\
&= pr_{T_{10}}\left(e^{-2f}[T,X]\right) - 2ie^{-2f}\underbrace{[\delta f, X]}_{=\nabla^W_{\delta f}X - \nabla^W_X \delta f} + X(2ie^{-2f})\delta f \\
&\quad + \underbrace{pr_{T_{10}}\left(2ie^{-2f}[\overline{\delta f}, X]\right)}_{=:2ie^{-2f}\nabla^W_{\overline{\delta f}}X}.
\end{aligned}
$$

We use that the Reeb vector field is parallel with respect to the Tanaka Webster connection and the torsion $Tor^W(T, X)$ is in $\Gamma(\overline{T_{10}})$. So $pr_{T_{10}}(e^{-2f}[T,X]) = e^{-2f}\nabla^W_T X$ and according to the definition of ∇^W we get

$$
\begin{aligned}
\hat\nabla^W_{\hat T} X &= \underline{e^{-2f}\nabla^W_T X - 2ie^{-2f}\nabla^W_{\delta f} X} + 2ie^{-2f}\nabla^W_X \delta f - 4ie^{-2f}X(f)\delta f + \underline{2ie^{-2f}\nabla^W_{\overline{\delta f}} X} \\
&= \underline{\nabla^W_{e^{-2f}T - 2ie^{-2f}\delta f + 2ie^{-2f}\overline{\delta f}}} X + 2ie^{-2f}\nabla^W_X \delta f - 4ie^{-2f}X(f)\delta f \\
&= \underline{\nabla^W_{\hat T}} X + 2ie^{-2f}\nabla^W_X \delta f - 4ie^{-2f}X(f)\delta f.
\end{aligned}
$$

With the formulas already obtained we can easily determine the covariant derivative in direction of the Reeb vector field T of the original pseudo-hermitian structure θ.

$$\begin{aligned}
\hat{\nabla}^W_T X &= \hat{\nabla}^W_{e^{2f}\hat{T}+2i\delta f-2i\overline{\delta f}} X \\
&= e^{2f}\left(\nabla^W_{\hat{T}} X + 2ie^{-2f}\nabla^W_X \delta f - 4ie^{-2f} X(f)\delta f\right) \\
&\quad +2i\hat{\nabla}^W_{\delta f} X - 2i\hat{\nabla}^W_{\overline{\delta f}} X \\
&= \underline{e^{2f}\nabla^W_{\hat{T}} X} + 2i\nabla^W_X \delta f \underbrace{-4iX(f)\delta f} \\
&\quad +2i\left(\underline{\nabla^W_{\delta f} X} + 2\delta f(f)X + \underbrace{2X(f)\delta f}\right) - 2i\left(\underline{\nabla^W_{\overline{\delta f}} X} - 2L_\theta(X,\delta f)\delta f\right)
\end{aligned}$$

The underlined terms can be combined to $\nabla^W_T X$ and the underbraced terms cancel out.

$$\begin{aligned}
\hat{\nabla}^W_T X &= \underline{\nabla^W_T X} + 2i\nabla^W_X \delta f + 4i\delta f(f)X + 4i \underbrace{L_\theta(X,\delta f)}_{=\sum_\alpha L_\theta(X,\overline{Z_\alpha(f)}Z_\alpha)} \delta f \\
&= \nabla^W_T X + 2i\nabla^W_X \delta f + 4i\delta f(f)X + 4i \underbrace{\sum_\alpha Z_\alpha(f) L_\theta(X, Z_\alpha)}_{=X(f)} \delta f \\
&= \nabla^W_T X + 2i\nabla^W_X \delta f + 4i\delta f(f)X + 4iX(f)\delta f
\end{aligned}$$

For the covariant derivative of a section $Y \in \Gamma(T_{10})$ in direction of a vector field in the complexified tangent space $X = pr_{T_{10}}X + pr_{\overline{T_{10}}}X + \theta(X)T \in \Gamma(TM^{\mathbb{C}})$ we get:

$$\begin{aligned}
\hat{\nabla}^W_X Y &= \hat{\nabla}^W_{pr_{T_{10}}X} Y + \hat{\nabla}^W_{pr_{\overline{T_{10}}}X} Y + \theta(X)\hat{\nabla}^W_T Y \\
&= \nabla^W_{pr_{T_{10}}X} Y + 2\left(pr_{T_{10}}X\right)(f)Y + 2Y(f)pr_{T_{10}}X \\
&\quad +\nabla^W_{pr_{\overline{T_{10}}}X} Y - 2L_\theta\left(Y, \overline{pr_{\overline{T_{10}}}X}\right)\delta f \\
&\quad +\theta(X)\left(\nabla^W_T Y + 2i\nabla^W_Y \delta f + 4i\delta f(f)Y + 4iY(f)\delta f\right) \\
&= \nabla^W_X Y + 2\left(pr_{T_{10}}X\right)(f)Y + 2Y(f)pr_{T_{10}}X \\
&\quad -2\underbrace{L_\theta\left(Y, pr_{T_{10}}\overline{X}\right)}_{=L_\theta(Y,\overline{X})}\delta f + 2i\theta(X)\left(\nabla^W_Y \delta f + 2\delta f(f)Y + 2Y(f)\delta f\right).
\end{aligned}$$

We present our results in the following lemma.

Lemma 2.9 *Let (M, T_{10}, θ) be a strictly pseudo-convex CR manifold and $\hat{\theta} = e^{2f}\theta$ another pseudo-hermitian form, $f \in \mathcal{C}^\infty(M)$. Then we have with $\delta f := \sum_\alpha \overline{Z_\alpha}(f)Z_\alpha \in \Gamma(T_{10})$ and $\hat{T}$ denoting the Reeb vector field with respect to $\hat{\theta}$:*

$$\hat{T} = e^{-2f}(T - 2i\delta f + 2i\overline{\delta f}).$$

For the Levi form $L_{\hat{\theta}}$ it holds

$$\begin{aligned}
L_{\hat{\theta}} &= e^{2f}L_\theta + e^{2f}\sum_\alpha \Big(2i\big(df(\overline{Z_\alpha})\theta^{\overline{\alpha}} - df(Z_\alpha)\theta^\alpha\big)\otimes\overline{\theta} \\
&\qquad +2i\theta\otimes\left(df(\overline{Z_\alpha})\overline{\theta^{\alpha}} - df(Z_\alpha)\overline{\theta^{\overline{\alpha}}}\right) \\
&\qquad +8Z_\alpha(f)\overline{Z_\alpha}(f)\theta\otimes\overline{\theta}\Big)
\end{aligned}$$

and the Tanaka Webster connection is given by

$$\begin{aligned}
\hat{\nabla}^W_X Y &= \nabla^W_X Y + 2\left(pr_{T_{10}}X\right)(f)Y + 2Y(f)pr_{T_{10}}X \\
&\quad -2L_\theta\left(Y, pr_{T_{10}}\overline{X}\right)\delta f + 2i\theta(X)\left(\nabla^W_Y \delta f + 2\delta f(f)Y + 2Y(f)\delta f\right),
\end{aligned}$$

where Y is a section in T_{10} and $X = pr_{T_{10}}X + pr_{\overline{T_{10}}}X + \theta(X)T \in \Gamma(TM^{\mathbb{C}})$ is some vector field in the complexified tangent space.

Later on we will need to know how the Webster scalar curvature transforms.

Lemma 2.10 *Let (M, T_{10}, θ) be a strictly pseudo-convex CR manifold and $\hat{\theta} = e^{2f}\theta$ another pseudo-hermitian form, $f \in \mathcal{C}^\infty(M)$. Then we have for the Webster scalar curvature:*

$$\hat{R}^W = e^{-2f}\Big(R^W - 2(n+1)\sum_k \Big(\big(\nabla^W_{Z_k} df\big)(\overline{Z_k}) + \big(\nabla^W_{\overline{Z_k}} df\big)(Z_k)\Big) - 4n(n+1)\delta f(f)\Big)$$

Proof: To calculate the Webster scalar curvature for $\hat{\theta}$ we use the transformation formulas of the previous lemma. Several terms cancel out immediately since the vector fields inserted have no component in direction of the Reeb vector field T.

$$\begin{aligned}
\hat{R}^W &= \sum_{lk} \hat{\mathcal{R}}^W\left(\hat{Z}_l, \overline{\hat{Z}_l}, \hat{Z}_k, \overline{\hat{Z}_k}\right) \\
&= e^{-4f} \sum_{lk} \hat{\mathcal{R}}^W\left(Z_l, \overline{Z_l}, Z_k, \overline{Z_k}\right) \\
&= e^{-4f} \sum_{lk} L_{\hat{\theta}}\left(\hat{\nabla}^W_{Z_l}\hat{\nabla}^W_{\overline{Z_l}} Z_k - \hat{\nabla}^W_{\overline{Z_l}}\hat{\nabla}^W_{Z_l} Z_k - \hat{\nabla}^W_{[Z_l,\overline{Z_l}]} Z_k, Z_k\right) \\
&= e^{-2f} \sum_{lk} L_{\theta}\left(\hat{\nabla}^W_{Z_l}\hat{\nabla}^W_{\overline{Z_l}} Z_k - \hat{\nabla}^W_{\overline{Z_l}}\hat{\nabla}^W_{Z_l} Z_k - \hat{\nabla}^W_{[Z_l,\overline{Z_l}]} Z_k, Z_k\right) \\
&= e^{-2f} \sum_{lk} L_{\theta}\Big(\hat{\nabla}^W_{Z_l}\big(\nabla^W_{\overline{Z_l}} Z_k - \underbrace{2L_\theta(Z_k, Z_l)}_{=const.}\delta f\big) \\
&\qquad -\hat{\nabla}^W_{\overline{Z_l}}\left(\nabla^W_{Z_l} Z_k + 2df(Z_k)Z_l + 2df(Z_l)Z_k\right) \\
&\qquad -\nabla^W_{[Z_l,\overline{Z_l}]} Z_k - 2\big(\underbrace{pr_{T_{10}}[Z_l,\overline{Z_l}]}_{=-\nabla^W_{\overline{Z_l}} Z_l}\big)(f) Z_k \\
&\qquad -2Z_k(f)\underbrace{pr_{T_{10}}[Z_l,\overline{Z_l}]}_{=-\nabla^W_{\overline{Z_l}} Z_l} + 2L_\theta\big(Z_k, \overline{[Z_l,\overline{Z_l}]}\big)\delta f \\
&\qquad -2i\ \underbrace{\theta\big([Z_l,\overline{Z_l}]\big)}_{=-iL_\theta(Z_l,Z_l)}\ \big(\nabla^W_{Z_k}\delta f + 2\delta f(f)Z_k + 2Z_k(f)\delta f\big), Z_k\Big) \\
&= e^{-2f} \sum_{lk} L_{\theta}\Big(\hat{\nabla}^W_{Z_l}\big(\nabla^W_{\overline{Z_l}} Z_k - 2\delta_{kl}\delta f\big) \\
&\qquad -\hat{\nabla}^W_{\overline{Z_l}}\left(\nabla^W_{Z_l} Z_k + 2df(Z_k)Z_l + 2df(Z_l)Z_k\right) \\
&\qquad -\nabla^W_{[Z_l,\overline{Z_l}]} Z_k + 2\big(\nabla^W_{\overline{Z_l}} Z_l\big)(f) Z_k + 2Z_k(f)\nabla^W_{\overline{Z_l}} Z_l \\
&\qquad +2L_\theta\big(Z_k, \overline{[Z_l,\overline{Z_l}]}\big)\delta f - 2\nabla^W_{Z_k}\delta f - 4\delta f(f)Z_k - 4Z_k(f)\delta f, Z_k\Big) \\
&= e^{-2f} \sum_{lk} L_{\theta}\Big(\underline{\nabla^W_{Z_l}\nabla^W_{\overline{Z_l}} Z_k} + 2df(Z_l)\nabla^W_{\overline{Z_l}} Z_k + 2df\left(\nabla^W_{\overline{Z_l}} Z_k\right) Z_l \\
&\qquad -2\delta_{kl}\left(\nabla^W_{Z_l}\delta f + 2df(Z_l)\delta f + 2\delta f(f)Z_l\right) \\
&\qquad \underline{-\nabla^W_{\overline{Z_l}}\nabla^W_{Z_l} Z_k} + 2L_\theta\left(\nabla^W_{Z_l} Z_k, Z_l\right)\delta f \\
&\qquad -\overline{Z_l}\left(2df(Z_k)\right) Z_l - 2df(Z_k)\nabla^W_{\overline{Z_l}} Z_l + 4df(Z_k)L_\theta(Z_l, Z_l)\delta f \\
&\qquad -\overline{Z_l}\left(2df(Z_l)\right) Z_k - 2df(Z_l)\nabla^W_{\overline{Z_l}} Z_k + 4df(Z_l)L_\theta(Z_k, Z_l)\delta f \\
&\qquad \underline{-\nabla^W_{[Z_l,\overline{Z_l}]} Z_k} + 2L_\theta\left(Z_k, \overline{\nabla^W_{Z_l}\overline{Z_l}}\right)\delta f \\
&\qquad +2df(Z_k)\nabla^W_{\overline{Z_l}} Z_l + 2df\left(\nabla^W_{\overline{Z_l}} Z_l\right) Z_k \\
&\qquad -2\nabla^W_{Z_k}\delta f - 4\delta f(f)Z_k - 4df(Z_k)\delta f, Z_k\Big)
\end{aligned}$$

The underlined terms form the Webster scalar curvature of the original pseudo-hermitian form θ.

$$
\begin{aligned}
\hat{R}^W \;=\;& e^{-2f}R^W \\
&+e^{-2f}\sum_{lk}\Big(\overbrace{2df(Z_l)L_\theta\big(\nabla^W_{\overline{Z_l}}Z_k, Z_k\big)}+2\delta_{kl}df\left(\nabla^W_{\overline{Z_l}}Z_k\right) \\
&\qquad -2\delta_{kl}L_\theta\big(\nabla^W_{Z_l}\delta f, Z_k\big)\underbrace{-4\delta_{kl}df(Z_l)L_\theta\big(\delta f, Z_k\big)}-4\delta_{kl}\delta f(f) \\
&\qquad +2L_\theta\left(\nabla^W_{Z_l}Z_k, Z_l\right)L_\theta\big(\delta f, Z_k\big) \\
&\qquad -2\delta_{kl}\overline{Z_l}\left(df(Z_k)\right)-\big\{\overbrace{2df(Z_k)L_\theta\big(\nabla^W_{\overline{Z_l}}Z_l, Z_k\big)\big\}}+\underline{4df(Z_k)L_\theta\big(\delta f, Z_k\big)} \\
&\qquad -2\overline{Z_l}\left(df(Z_l)\right)\overbrace{-2df(Z_l)L_\theta\big(\nabla^W_{\overline{Z_l}}Z_k, Z_k\big)}\underbrace{+4\delta_{kl}df(Z_l)L_\theta\big(\delta f, Z_k\big)} \\
&\qquad +2L_\theta(Z_k, \nabla^W_{\overline{Z_l}}Z_l)L_\theta\big(\delta f, Z_k\big) \\
&\qquad +\big\{2df(Z_k)L_\theta\big(\nabla^W_{\overline{Z_l}}Z_l, Z_k\big)\big\}+2df\left(\nabla^W_{\overline{Z_l}}Z_l\right) \\
&\qquad -2L_\theta\big(\nabla^W_{Z_k}\delta f, Z_k\big)-4\delta f(f)\underline{-4df(Z_k)L_\theta\big(\delta f, Z_k\big)}\Big)
\end{aligned}
$$

The marked terms cancel out and we obtain:

$$
\begin{aligned}
\hat{R}^W \;=\;& e^{-2f}R^W \\
&+e^{-2f}\sum_{lk}\Big(2\delta_{kl}df\left(\nabla^W_{\overline{Z_l}}Z_k\right)-2\delta_{kl}L_\theta\big(\nabla^W_{Z_l}\delta f, Z_k\big)-4\delta_{kl}\delta f(f) \\
&\qquad +2L_\theta\left(\nabla^W_{Z_l}Z_k, Z_l\right)L_\theta\big(\delta f, Z_k\big)-2\delta_{kl}\overline{Z_l}\left(df(Z_k)\right)-2\overline{Z_l}\left(df(Z_l)\right) \\
&\qquad +2L_\theta(Z_k, \nabla^W_{\overline{Z_l}}Z_l)L_\theta\big(\delta f, Z_k\big)+2df\left(\nabla^W_{\overline{Z_l}}Z_l\right) \\
&\qquad -2L_\theta\big(\nabla^W_{Z_k}\delta f, Z_k\big)-4\delta f(f)\Big).
\end{aligned}
$$

Recall the definition of $\delta f := \sum_\alpha \overline{Z_\alpha}(f)Z_\alpha$. Hence we have $L_\theta(\delta f, Z_k) = \overline{Z_k}(f)$.

$$
\begin{aligned}
\hat{R}^W \;=\;& e^{-2f}R^W \\
&+e^{-2f}\sum_{lk}\Big(2\delta_{kl}df\left(\nabla^W_{\overline{Z_l}}Z_k\right)-2\delta_{kl}L_\theta\big(\nabla^W_{Z_l}\delta f, Z_k\big) \\
&\qquad +2L_\theta\left(\nabla^W_{Z_l}Z_k, Z_l\right)\overline{Z_k}(f)-2\delta_{kl}\overline{Z_l}\left(df(Z_k)\right)-2\overline{Z_l}\left(df(Z_l)\right) \\
&\qquad +2L_\theta(Z_k, \nabla^W_{\overline{Z_l}}Z_l)\overline{Z_k}(f)+2df\left(\nabla^W_{\overline{Z_l}}Z_l\right) \\
&\qquad -2L_\theta\big(\nabla^W_{Z_k}\delta f, Z_k\big)-4(1+\delta_{kl})\delta f(f)\Big) \\
\;=\;& e^{-2f}R^W \\
&+e^{-2f}\Big(2\textstyle\sum_k df\left(\nabla^W_{\overline{Z_k}}Z_k\right)-2\sum_k L_\theta\big(\nabla^W_{Z_k}\delta f, Z_k\big) \\
&\qquad +2\textstyle\sum_{lk}\big(\underbrace{L_\theta\left(\nabla^W_{Z_l}Z_k, Z_l\right)+L_\theta(Z_k, \nabla^W_{\overline{Z_l}}Z_l)}_{=Z_l(L_\theta(Z_k,Z_l))=0}\big)\overline{Z_k}(f) \\
&\qquad -2(n+1)\textstyle\sum_k \overline{Z_k}\left(df(Z_k)\right)+2n\sum_k df\left(\nabla^W_{\overline{Z_k}}Z_k\right) \\
&\qquad -2n\textstyle\sum_k L_\theta\big(\nabla^W_{Z_k}\delta f, Z_k\big)-4n(n+1)\delta f(f)\Big)
\end{aligned}
$$

We combine several sums.

$$
\begin{aligned}
\hat{R}^W &= e^{-2f} R^W \\
&\quad + e^{-2f} \Big(2(n+1) \sum_k df \left(\nabla^W_{\overline{Z_k}} Z_k \right) - 2(n+1) \sum_k L_\theta \big(\nabla^W_{Z_k} \delta f, Z_k \big) \\
&\qquad\qquad -2(n+1) \sum_k \underbrace{\overline{Z_k} \left(df(Z_k) \right)}_{= \left(\nabla^W_{\overline{Z_k}} df \right)(Z_k) + df\left(\nabla^W_{\overline{Z_k}} Z_k \right)} -4n(n+1)\delta f(f) \Big) \\
&= e^{-2f} R^W \\
&\quad + e^{-2f} \Big(-2(n+1) \sum_k L_\theta \big(\nabla^W_{Z_k} \delta f, Z_k \big) \\
&\qquad\qquad -2(n+1) \sum_k \big(\nabla^W_{\overline{Z_k}} df \big)(Z_k) - 4n(n+1)\delta f(f) \Big)
\end{aligned}
$$

Let us take an extra look at the term $\sum_k L_\theta \big(\nabla^W_{Z_k} \delta f, Z_k \big)$. We have

$$
\begin{aligned}
\sum_k L_\theta \big(\nabla^W_{Z_k} \delta f, Z_k \big) &= \sum_k L_\theta \left(\nabla^W_{Z_k} \big(\sum_l df(\overline{Z_l}) Z_l \big), Z_k \right) \\
&= \sum_{kl} L_\theta \left(Z_k \big(df(\overline{Z_l}) \big) Z_l + df(\overline{Z_l}) \nabla^W_{Z_k} Z_l, Z_k \right) \\
&= \sum_k Z_k \big(df(\overline{Z_k}) \big) + \sum_{kl} df(\overline{Z_l}) \underbrace{L_\theta \left(\nabla^W_{Z_k} Z_l, Z_k \right)}_{= -L_\theta \left(Z_l, \nabla^W_{\overline{Z_k}} Z_k \right)} \\
&= \sum_k Z_k \big(df(\overline{Z_k}) \big) - \sum_k df \left(\sum_l \overline{L_\theta \big(\nabla^W_{\overline{Z_k}} Z_k, Z_l \big) Z_l} \right) \\
&= \sum_k Z_k \big(df(\overline{Z_k}) \big) - \sum_k df \left(\nabla^W_{Z_k} \overline{Z_k} \right) \\
&= \sum_k \big(\nabla^W_{Z_k} df \big)(\overline{Z_k}).
\end{aligned}
$$

Hence we can write

$$
\hat{R}^W = e^{-2f} \left(R^W - 2(n+1) \sum_k \Big(\big(\nabla^W_{Z_k} df \big)(\overline{Z_k}) + \big(\nabla^W_{\overline{Z_k}} df \big)(Z_k) \Big) - 4n(n+1)\delta f(f) \right).
$$

□

The formulas proved in this section will now be used to construct the Fefferman space.

2.2 The Fefferman Space According to [BL04]

In this section we will describe the construction of the Fefferman space of a strictly pseudo-convex CR manifold according to [Bau99] and [BL04]. This will be achieved by constructing a conformal class with the help of the Levi form and a connection derived from the Tanaka Webster connection. Using the transformation formulas of ∇^W and R^W will yield the independence of this conformal class from the chosen pseudo-hermitian form θ. Hence the Fefferman space will be CR invariant.

Definition 2.12 *The canonical complex line bundle of a strictly pseudo-convex CR manifold* (M^{2n+1}, H, θ) *is given by*

$$
\mathcal{K} := \left\{ \omega \in \textstyle\bigwedge^{n+1} (TM^{\mathbb{C}})^* \;\middle|\; i_V \omega = 0 \textit{ for all } V \in \overline{T_{10}} \right\}.
$$

In order to give a local basis in $\mathcal{K}$ we choose at first a local unitary basis $(Z_1, \dots, Z_n)$ in (T_{10}, L_θ). This is enlarged to form a local basis $(Z_1, \dots, Z_n, \overline{Z_1}, \dots, \overline{Z_n}, T)$ in the complexified tangent space $TM^{\mathbb{C}}$. We denote the corresponding dual basis of $(TM^{\mathbb{C}})^*$ with $(\theta^1, \dots, \theta^n, \overline{\theta^1}, \dots, \overline{\theta^n}, \theta)$. Hence $\widehat{\tau} := \theta \wedge \theta^1 \wedge \dots \wedge \theta^n$ is a local basis in $\mathcal{K}$.

With $\mathcal{K}^* := \mathcal{K} \setminus \{0\}$ we obtain the S^1-principal bundle

$$\mathcal{F} := \mathcal{K}^*/_{\mathbb{R}^+} \longrightarrow M$$

and $\tau := \frac{\widehat{\tau}}{|\widehat{\tau}|}$ is a local section in $\mathcal{F} \longrightarrow M$.
Now we are looking for a conformal structure on $\mathcal{F}$, uniquely defined by the CR manifold (M, H, J). To find this structure let $A \in \mathcal{C}(\mathcal{F})$ be a connection. Using the horizontal lifts corresponding to A, which we will denote by *, we get the following splitting of the tangent space of $\mathcal{F}$:

$$\begin{array}{ccclc}
\overset{S^1}{\circlearrowleft}\mathcal{F} & \longleftarrow & T_x\mathcal{F} & = Th_x^A\mathcal{F} & \oplus Tv_x\mathcal{F} \\
 & & & = H_x^* \oplus (\mathbb{R}T_x)^* & \oplus Tv_x\mathcal{F} \\
\Big\downarrow \pi & & \Big\downarrow d\pi_x & & \\
M & \longleftarrow & T_xM & = H_x \oplus \mathbb{R}T_x &
\end{array}$$

Let us take a look at the following metric on $\mathcal{F}$ with $c \in \mathbb{R}$ fix.

$$h_{A,c} := \pi^* L_\theta - 2ci(\pi^*\theta) \odot A$$

where $\odot$ denotes the symmetric tensor product. Consequently H^* and $span(T^*, Tv\mathcal{F})$ are orthogonal with respect to $h_{A,c}$ and $\pi^* L_\theta$ is riemannian on H^* and vanishes everywhere else. For $N := \widetilde{i}$, the fundamental vector field of $i \in LA(S^1) = i\mathbb{R}$ and T^*, the horizontal lift of the Reeb vector field T we have:

$$\begin{aligned}
(\pi^*\theta \odot A)(N, N) &= \theta(\underbrace{d\pi N}_{=0}) \cdot \underbrace{A(N)}_{=i} \\
&= 0, \\
(\pi^*\theta \odot A)(T^*, T^*) &= \underbrace{\theta(T)}_{=1} \cdot \underbrace{A(T^*)}_{=0} \\
&= 0 \text{ and} \\
(\pi^*\theta \odot A)(T^*, N) &= \tfrac{1}{2}\big(\underbrace{\theta(T)}_{=1} \cdot \underbrace{A(N)}_{=i} + \theta(\underbrace{d\pi N}_{=0}) \cdot \underbrace{A(T^*)}_{=0}\big) \\
&= \tfrac{i}{2}.
\end{aligned}$$

Hence we get

$$\begin{aligned}
h_{A,c}(N, N) &= 0, \\
h_{A,c}(T^*, T^*) &= 0 \text{ and} \\
h_{A,c}(T^*, N) &= -2ci(\pi^*\theta \odot A)(T^*, N) \\
&= c.
\end{aligned}$$

$h_{A,c}$ is therefore a Lorentzian metric on the S^1-principal bundle $\mathcal{F}$. For every element $\alpha \in S^1$ the action $R_\alpha : \mathcal{F} \longrightarrow \mathcal{F}$ is an isometry since S^1 is abelian and

$$\begin{aligned}
R_\alpha^* h_{A,c} &= R_\alpha^*(\pi^* L_\theta - 2ci(\pi^*\theta) \odot A) \\
&= \pi^* L_\theta - 2ci(\pi^*\theta) \odot R_\alpha^* A \\
&= \pi^* L_\theta - 2ci(\pi^*\theta) \odot Ad(\alpha^{-1}) \circ A \\
&\overset{S^1 \text{ ab.}}{=} \pi^* L_\theta - 2ci(\pi^*\theta) \odot A \\
&= h_{A,c}.
\end{aligned}$$

Consequently the fundamental vector field $N = \widetilde{i}$ generated by i is a light like Killing vector field.

With the help of the Tanaka Webster connection $\nabla^W : \Gamma(TM^{\mathbb{C}}) \longrightarrow \Gamma\big((TM^{\mathbb{C}})^* \otimes TM^{\mathbb{C}}\big)$ we will now specify a connection A^W on the S^1-principal bundle $\mathcal{F}$ and a number $c \in \mathbb{R}$, such that the conformal class $[h_{A,c}]$ is independent of the choice of θ.

At first we will describe the connection A^W locally.

Let $s = (Z_1, \ldots, Z_n) : U \longrightarrow T_{10}^n$ be a local unitary frame with respect to the Levi form L_θ for the open subset $U \subset M$. Then $\tau_s := [\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n] : U \longrightarrow \mathcal{F}$ is a local section in $\mathcal{F}$, where θ^k is dual to Z_k. We can write

$$\nabla^W Z_\alpha = \sum_{\beta=1}^{n} \omega_{\alpha\beta} \otimes Z_\beta \in \Gamma\left((TM^{\mathbb{C}})^* \otimes T_{10}\right)$$

and obtain the matrix $\omega_s := \big(\omega_{\alpha\beta}\big)$, the local connection form of ∇^W associated to the section s. Hence the Tanaka Webster connection ∇^W defines a covariant derivative $\nabla^{\mathcal{K}}$ in $\mathcal{K}$ via

$$\begin{aligned}
\nabla^{\mathcal{K}} \widehat{\tau}_s &= \nabla^{\mathcal{K}}(\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n) \\
&:= \underbrace{\nabla^W \theta}_{:=0} \wedge \theta^1 \wedge \cdots \wedge \theta^n + \theta \wedge \underbrace{\nabla^W \theta^1}_{:=-\sum_\alpha \omega_{1\alpha}\theta^\alpha} \wedge \cdots \wedge \theta^n + \cdots + \theta \wedge \theta^1 \wedge \cdots \wedge \underbrace{\nabla^W \theta^n}_{:=-\sum_\alpha \omega_{n\alpha}\theta^\alpha} \\
&= -\sum_{\alpha=1}^{n} \omega_{\alpha\alpha} \theta \wedge \theta^1 \wedge \cdots \wedge \theta^n \\
&= -Tr(\omega_s) \cdot \widehat{\tau}_s.
\end{aligned}$$

Since ∇^W is metric with respect to the Levi form L_θ, we have for all vector fields $X \in \mathfrak{X}(M)$

$$\begin{aligned}
0 &= X\big(\underbrace{L_\theta(Z_\alpha, Z_\alpha)}_{\equiv 1}\big) \\
&= L_\theta\big(\nabla^W_X Z_\alpha, Z_\alpha\big) + L_\theta\big(Z_\alpha, \nabla^W_X Z_\alpha\big) \\
&= \omega_{\alpha\alpha}(X) + \overline{\omega_{\alpha\alpha}(X)} \\
&= 2Re(\omega_{\alpha\alpha}(X)).
\end{aligned}$$

Consequently $\omega_{\alpha\alpha}(X)$ is purely imaginary as is the trace $Tr(\omega_s(X)) \in C^\infty(U, i\mathbb{R})$ for all vector fields $X \in \mathfrak{X}(M)$. For every local section s in (T_{10}, L_θ) we therefore get a local 1-form on M with values in $i\mathbb{R} = LA(S^1)$

$$\begin{aligned}
A_s^W : U &\longrightarrow i\mathbb{R} \\
X &\mapsto -Tr(\omega_s(X)).
\end{aligned}$$

This family $\{A_s^W, \tau_s\}_{s:U \longrightarrow (T_{10}, L_\theta)}$ yields a S^1-principal bundle connection $A^W : T\mathcal{F} \longrightarrow i\mathbb{R}$ on $\mathcal{F}$ with $(\tau_s)^* A^W = A_s^W$, since this is generated by a covariant derivative.

Lemma 2.11 *The curvature of the S^1-principal bundle connection A^W defined by the Tanaka Webster connection satisfies*

$$\Omega^{A^W} \in \Omega^2(M, i\mathbb{R}).$$

More precisely the local curvature form holds $\Omega_s^{A^W} = -Ric^W$.

Proof: Since the Lie group S^1 is abelian, the horizontal, Ad-invariant forms with values in $i\mathbb{R} = LA(S^1)$ can be seen as forms on the base manifold M with values in $i\mathbb{R}$. We can write as well $\Omega^{A^W} \in \Omega^2_{\text{hor}}(\mathcal{F}, i\mathbb{R})^{Ad} \simeq \Omega^2(M, i\mathbb{R})$.

Let $X, Y \in \mathfrak{X}(M)$ be vector fields on M. We have for the Webster Ricci curvature:

$$
\begin{aligned}
Ric^W(X,Y) &= \sum_\alpha L_\theta\Big(([\nabla^W_X, \nabla^W_Y] - \nabla^W_{[X,Y]})Z_\alpha, Z_\alpha\Big) \\
&= \sum_\alpha L_\theta\Big(\nabla^W_X\big(\sum_\beta \omega_{\alpha\beta}(Y)Z_\beta\big) - \nabla^W_Y\big(\sum_\beta \omega_{\alpha\beta}(X)Z_\beta\big) \\
&\qquad\qquad -\big(\sum_\beta \omega_{\alpha\beta}([X,Y])Z_\beta\big), Z_\alpha\Big) \\
&= \sum_{\alpha\beta} L_\theta\Big(X\left(\omega_{\alpha\beta}(Y)\right)Z_\beta + \underline{\omega_{\alpha\beta}(Y)\sum_\gamma \omega_{\beta\gamma}(X)Z_\gamma} \\
&\qquad\qquad -Y\left(\omega_{\alpha\beta}(X)\right)Z_\beta \underline{-\omega_{\alpha\beta}(X)\sum_\gamma \omega_{\beta\gamma}(Y)Z_\gamma} \\
&\qquad\qquad -\omega_{\alpha\beta}([X,Y])Z_\beta, Z_\alpha\Big).
\end{aligned}
$$

The underlined terms cancel out.

$$
\begin{aligned}
Ric^W(X,Y) &= \sum_\alpha \Big(X\left(\omega_{\alpha\alpha}(Y)\right) - Y\left(\omega_{\alpha\alpha}(X)\right) - \omega_{\alpha\alpha}([X,Y])\Big) \\
&= \sum_\alpha d\omega_{\alpha\alpha}(X,Y) \\
&= Tr(d\omega_s)(X,Y)
\end{aligned}
$$

Since $A^W_s = -Tr(\omega_s)$ is true we obtain the result wanted

$$\Omega^{A^W}_s = dA^W_s = -Tr(d\omega_s) = -Ric^W.$$

$\square$

All other connections $A \in \mathcal{C}(\mathcal{F})$ are of the shape $A = A^W - i\omega$, where ω is a 1-form on M with real values, $i\omega \in \Omega^1(M, i\mathbb{R}) \simeq \Omega^1_{\text{hor}}(\mathcal{F}, i\mathbb{R})^{Ad}$.
Now we define

$$A^\theta := A^W - \frac{i}{2(n+1)} R^W \cdot \theta \in \mathcal{C}(\mathcal{F}).$$

Lemma 2.12 *With $c = \frac{2}{n+2}$ the conformal class of the metric $h_\theta := h_{A^\theta, c}$,*

$$h_\theta = \pi^* L_\theta - i\frac{4}{n+2}\pi^*\theta \odot A^\theta,$$

is CR invariant, i.e. $[h_\theta] = [h_{f\cdot\theta}]$.

Proof: Let (M, T_{10}, θ) be a strictly pseudo-convex CR manifold and $\hat{\theta} = e^{2f}\theta$ another pseudo-hermitian form, $f \in \mathcal{C}^\infty(M)$. According to Lemma 2.9 we have for the Reeb vector field $\hat{T} = e^{-2f}(T - 2i\delta f + 2i\overline{\delta f})$, where $\delta f := \sum_\alpha \overline{Z_\alpha}(f) Z_\alpha \in \Gamma(T_{10})$. The Levi form fulfills

$$
\begin{aligned}
L_{\hat{\theta}} &= e^{2f} L_\theta + e^{2f} \sum_\alpha \Big(2i\big(df(\overline{Z_\alpha})\theta^{\overline{\alpha}} - df(Z_\alpha)\theta^\alpha\big) \otimes \overline{\theta} \\
&\qquad\qquad +2i\theta \otimes \big(df(\overline{Z_\alpha})\overline{\theta^{\alpha}} - df(Z_\alpha)\overline{\theta^{\overline{\alpha}}}\big) \\
&\qquad\qquad +8Z_\alpha(f)\overline{Z_\alpha}(f)\theta \otimes \overline{\theta}\Big).
\end{aligned}
$$

And for the Tanaka Webster connection we have

$$
\begin{aligned}
\hat{\nabla}^W_X Y &= \nabla^W_X Y + 2\left(pr_{T_{10}} X\right)(f) Y + 2Y(f) pr_{T_{10}} X \\
&\quad -2L_\theta\left(Y, \overline{X}\right)\delta f + 2i\theta(X)\left(\nabla^W_Y \delta f + 2\delta f(f) Y + 2Y(f)\delta f\right),
\end{aligned}
$$

where Y is a section in T_{10} and $X = pr_{T_{10}} X + pr_{\overline{T_{10}}} X + \theta(X) T \in \Gamma(TM^{\mathbb{C}})$ is some vector field in the complexified tangent space.

We have to find the conformal factor between h_θ and $h_{\hat\theta}$. Hence we have to give formulas for the components of the metric $h_{f\theta}$ first. To compute $A^{f\theta}$ we write the coefficients $\hat\omega_{\alpha\beta}$ of $\hat\nabla^W$ in terms of θ:

$$\begin{aligned}
\sum_\beta \hat\omega_{\alpha\beta}\otimes\hat Z_\beta &= \hat\nabla^W_\cdot \hat Z_\alpha \\
&= \nabla^W_\cdot \hat Z_\alpha + 2\left(pr_{T_{10}}\cdot\right)(f)\hat Z_\alpha \\
&\quad +2\hat Z_\alpha(f)pr_{T_{10}}\cdot -2L_\theta\big(\hat Z_\alpha,\bar\cdot\big)\delta f \\
&\quad +2i\theta(\cdot)\left(\nabla^W_{\hat Z_\alpha}\delta f + 2\delta f(f)\hat Z_\alpha + 2\hat Z_\alpha(f)\delta f\right) \\
&= e^{-f}\nabla^W_\cdot Z_\alpha - e^{-f}df\otimes Z_\alpha + 2df\circ\Big(\sum_\beta \theta^\beta\otimes Z_\beta\Big)\otimes\hat Z_\alpha \\
&\quad +2e^{-f}df(Z_\alpha)\Big(\sum_\beta\theta^\beta\otimes Z_\beta\Big) - 2e^{-f}L_\theta\left(Z_\alpha,\bar\cdot\right)\delta f \\
&\quad +2i\theta\otimes\left(\nabla^W_{\hat Z_\alpha}\delta f + 2\delta f(f)\hat Z_\alpha + 2\hat Z_\alpha(f)\delta f\right).
\end{aligned}$$

Recall the definition $\delta f := \sum_\beta \overline{Z_\beta}(f)Z_\beta$.

$$\begin{aligned}
\sum_\beta \hat\omega_{\alpha\beta}\otimes\hat Z_\beta &= e^{-f}\sum_\beta\omega_{\alpha\beta}\otimes Z_\beta - df\otimes\hat Z_\alpha + 2\sum_\beta df(Z_\beta)\theta^\beta\otimes\hat Z_\alpha \\
&\quad +2\sum_\beta df(Z_\alpha)\theta^\beta\otimes\hat Z_\beta - 2e^{-f}\theta^{\overline\alpha}\otimes\delta f \\
&\quad +2i\theta\otimes\Big(\nabla^W_{\hat Z_\alpha}\big(\sum_\beta\overline{Z_\beta}(f)Z_\beta\big) + 2\delta f(f)\hat Z_\alpha + 2\hat Z_\alpha(f)\sum_\beta\overline{Z}_\beta(f)Z_\beta\Big) \\
&= \sum_\beta\omega_{\alpha\beta}\otimes\hat Z_\beta - df\otimes\hat Z_\alpha + 2\sum_\beta df(Z_\beta)\theta^\beta\otimes\hat Z_\alpha \\
&\quad +2\sum_\beta df(Z_\alpha)\theta^\beta\otimes\hat Z_\beta - 2e^{-f}\theta^{\overline\alpha}\otimes\sum_\beta\overline{Z_\beta}(f)Z_\beta \\
&\quad +2i\theta\otimes\Big(e^{-f}\sum_\beta Z_\alpha\big(df(\overline{Z_\beta})\big)Z_\beta + e^{-f}\sum_\beta df(\overline{Z_\beta})\nabla^W_{Z_\alpha}Z_\beta \\
&\qquad\qquad +2\delta f(f)\hat Z_\alpha + 2Z_\alpha(f)\sum_\beta\overline{Z}_\beta(f)\hat Z_\beta\Big) \\
&= \sum_\beta\omega_{\alpha\beta}\otimes\hat Z_\beta \underbrace{-df(T)\theta\otimes\hat Z_\alpha - \sum_\beta df(\overline{Z_\beta})\theta^{\overline\beta}\otimes\hat Z_\alpha + \sum_\beta df(Z_\beta)\theta^\beta\otimes\hat Z_\alpha}_{=-df\otimes\hat Z_\alpha+2\sum_\beta df(Z_\beta)\theta^\beta\otimes\hat Z_\alpha} \\
&\quad +2\sum_\beta df(Z_\alpha)\theta^\beta\otimes\hat Z_\beta - 2\theta^{\overline\alpha}\otimes\sum_\beta\overline{Z_\beta}(f)\hat Z_\beta \\
&\quad +2i\theta\otimes\Big(e^{-f}\sum_\beta Z_\alpha\big(df(\overline{Z_\beta})\big)Z_\beta + e^{-f}\sum_\gamma df(\overline{Z_\gamma})\nabla^W_{Z_\alpha}Z_\gamma \\
&\qquad\qquad +2\delta f(f)\hat Z_\alpha + 2Z_\alpha(f)\sum_\beta\overline{Z}_\beta(f)\hat Z_\beta\Big)
\end{aligned}$$

At first we will take a separate look at a part of this sum.

$$\begin{aligned}
&-df(T)\theta\otimes\hat Z_\alpha + 2i\theta\otimes\Big(e^{-f}\sum_\beta Z_\alpha\big(df(\overline{Z_\beta})\big)Z_\beta + e^{-f}\sum_\gamma df(\overline{Z_\gamma})\nabla^W_{Z_\alpha}Z_\gamma\Big) \\
&= e^{-f}i\theta\otimes\Big(\sum_\beta\delta_{\alpha\beta}idf(T)\cdot Z_\beta + 2\sum_\beta\left(\nabla^W_{Z_\alpha}df\right)(\overline{Z_\beta})\cdot Z_\beta + 2\sum_\beta df\left(\nabla^W_{Z_\alpha}\overline{Z_\beta}\right)\cdot Z_\beta \\
&\qquad +2\sum_{\beta\gamma}df(\overline{Z_\gamma})\underbrace{L_\theta\big(\nabla^W_{Z_\alpha}Z_\gamma,Z_\beta\big)}_{=-L_\theta\left(Z_\gamma,\nabla^W_{\overline{Z_\alpha}}Z_\beta\right)}\cdot Z_\beta\Big)
\end{aligned}$$

We use that the Levi form is metric for sections in T_{10} and that $(Z_1, \ldots, Z_m)$ is a unitary basis in (T_{10}, L_θ).

$$
\begin{aligned}
&-df(T)\theta \otimes \hat{Z}_\alpha + 2i\theta \otimes \Big(e^{-f} \sum_\beta Z_\alpha\big(df(\overline{Z_\beta})\big) Z_\beta + e^{-f} \sum_\gamma df(\overline{Z_\gamma}) \nabla^W_{Z_\alpha} Z_\gamma\Big) \\
=\;& e^{-f} i\theta \otimes \Big(\sum_\beta \delta_{\alpha\beta} i df(T) \cdot Z_\beta + 2\sum_\beta \left(\nabla^W_{Z_\alpha} df\right)(\overline{Z_\beta}) \cdot Z_\beta + 2\sum_\beta df\left(\nabla^W_{Z_\alpha} \overline{Z_\beta}\right) \cdot Z_\beta \\
&\qquad \underbrace{-2\sum_{\beta\gamma} df(\overline{Z_\gamma}) \overline{\omega_{\beta\gamma}(\overline{Z_\alpha})} \cdot Z_\beta}_{=\sum_\beta df\left(\nabla^W_{Z_\alpha} Z_\beta\right) \cdot Z_\beta} \Big) \\
=\;& i\theta \otimes \sum_\beta \left(\delta_{\alpha\beta} i df(T) + 2\left(\nabla^W_{Z_\alpha} df\right)(\overline{Z_\beta}) + 2df\left(\nabla^W_{Z_\alpha}\overline{Z_\beta}\right) - 2df\left(\nabla^W_{Z_\alpha}\overline{Z_\beta}\right)\right) \cdot \hat{Z}_\beta \\
=\;& i\theta \otimes \sum_\beta \left(iL_\theta(Z_\alpha, Z_\beta) df(T) + 2\left(\nabla^W_{Z_\alpha} df\right)(\overline{Z_\beta})\right) \cdot \hat{Z}_\beta \\
=\;& i\theta \otimes \sum_\beta \Big(df\big(\underbrace{iL_\theta(Z_\alpha, Z_\beta) T}_{=Tor^W(Z_\alpha, \overline{Z_\beta})}\big) + 2\left(\nabla^W_{Z_\alpha} df\right)(\overline{Z_\beta})\Big) \cdot \hat{Z}_\beta \\
=\;& i\theta \otimes \sum_\beta \Big(df\big(\nabla^W_{Z_\alpha}\overline{Z_\beta} - \nabla^W_{\overline{Z_\beta}} Z_\alpha - [Z_\alpha, \overline{Z_\beta}]\big) + 2\left(\nabla^W_{Z_\alpha} df\right)(\overline{Z_\beta})\Big) \cdot \hat{Z}_\beta
\end{aligned}
$$

We have $0 = d(df)(Z_\alpha, \overline{Z_\beta}) = Z_\alpha\big(df(\overline{Z_\beta})\big) - \overline{Z_\beta}\big(df(Z_\alpha)\big) - df[Z_\alpha, \overline{Z_\beta}]$. Hence we can write:

$$
\begin{aligned}
&-df(T)\theta \otimes \hat{Z}_\alpha + 2i\theta \otimes \Big(e^{-f} \sum_\beta Z_\alpha\big(df(\overline{Z_\beta})\big) Z_\beta + e^{-f} \sum_\gamma df(\overline{Z_\gamma}) \nabla^W_{Z_\alpha} Z_\gamma\Big) \\
=\;& i\theta \otimes \sum_\beta \Big(\underline{df\big(\nabla^W_{Z_\alpha}\overline{Z_\beta}\big)} - df\big(\nabla^W_{\overline{Z_\beta}} Z_\alpha\big) \underline{- Z_\alpha\big(df(\overline{Z_\beta})\big)} + \overline{Z_\beta}\big(df(Z_\alpha)\big) + \underline{2\left(\nabla^W_{Z_\alpha} df\right)(\overline{Z_\beta})}\Big) \cdot \hat{Z}_\beta \\
=\;& i\theta \otimes \sum_\beta \Big(\underline{\left(\nabla^W_{Z_\alpha} df\right)(\overline{Z_\beta})} + \Big(\nabla^W_{\overline{Z_\beta}} df\Big)(Z_\alpha)\Big) \cdot \hat{Z}_\beta.
\end{aligned}
$$

With this information we can continue calculating the coefficients $\hat{\omega}_{\alpha\beta}$ of $\hat{\nabla}^W$.

$$
\begin{aligned}
\sum_\beta \hat{\omega}_{\alpha\beta} \otimes \hat{Z}_\beta \;=\;& \sum_\beta \omega_{\alpha\beta} \otimes \hat{Z}_\beta - \sum_\beta df(\overline{Z_\beta})\theta^{\overline{\beta}} \otimes \hat{Z}_\alpha + \sum_\beta df(Z_\beta)\theta^\beta \otimes \hat{Z}_\alpha \\
&+2\sum_\beta df(Z_\alpha)\theta^\beta \otimes \hat{Z}_\beta - 2\theta^{\overline{\alpha}} \otimes \sum_\beta df(\overline{Z_\beta})\hat{Z}_\beta \\
&+4i\theta \otimes \Big(\delta f(f)\hat{Z}_\alpha + df(Z_\alpha)\sum_\beta df(\overline{Z}_\beta)\hat{Z}_\beta\Big) \\
&+i\theta \otimes \sum_\beta \Big(\left(\nabla^W_{Z_\alpha} df\right)(\overline{Z_\beta}) + \Big(\nabla^W_{\overline{Z_\beta}} df\Big)(Z_\alpha)\Big) \cdot \hat{Z}_\beta \\
=\;& \sum_\beta \Big(\omega_{\alpha\beta} - \delta_{\alpha\beta}\sum_\gamma df(\overline{Z_\gamma})\theta^{\overline{\gamma}} + \delta_{\alpha\beta}\sum_\gamma df(Z_\gamma)\theta^\gamma \\
&\quad +2df(Z_\alpha)\theta^\beta - 2df(\overline{Z_\beta})\theta^{\overline{\alpha}} + 4i\delta_{\alpha\beta}\delta f(f)\theta + 4i df(Z_\alpha) df(\overline{Z}_\beta)\theta \\
&\quad +i\left(\nabla^W_{Z_\alpha} df\right)(\overline{Z_\beta})\theta + i\Big(\nabla^W_{\overline{Z_\beta}} df\Big)(Z_\alpha)\theta\Big) \cdot \hat{Z}_\beta
\end{aligned}
$$

We obtain:

$$
\begin{aligned}
\hat{\omega}_{\alpha\beta} \;=\;& \omega_{\alpha\beta} + 2\big(df(Z_\alpha)\theta^\beta - df(\overline{Z_\beta})\theta^{\overline{\alpha}}\big) + \delta_{\alpha\beta}\sum_\gamma \left(df(Z_\gamma)\theta^\gamma - df(\overline{Z_\gamma})\theta^{\overline{\gamma}}\right) \\
&+i\Big(\big(\nabla^W_{Z_\alpha} df\big)(\overline{Z_\beta}) + \big(\nabla^W_{\overline{Z_\beta}} df\big)(Z_\alpha) + 4df(Z_\alpha) df(\overline{Z}_\beta) + 4\delta_{\alpha\beta}\delta f(f)\Big)\theta.
\end{aligned}
$$

Consequently it holds

$$
\begin{aligned}
\hat{A}^W &= -Tr(\omega_{\hat{s}}) \\
&= -\sum_\alpha \hat{\omega}_{\alpha\alpha} \\
&= -\sum_\alpha \Big(\omega_{\alpha\alpha} + \underbrace{2\big(df(Z_\alpha)\theta^\alpha - df(\overline{Z_\alpha})\theta^{\overline{\alpha}}\big) + \sum_\gamma \big(df(Z_\gamma)\theta^\gamma - df(\overline{Z_\gamma})\theta^{\overline{\gamma}}\big)} \\
&\qquad + i\Big(\big(\nabla^W_{Z_\alpha} df\big)\,(\overline{Z_\alpha}) + \Big(\nabla^W_{\overline{Z_\alpha}} df\Big)\,(Z_\alpha) + \underline{4df(Z_\alpha)df(\overline{Z_\alpha}) + 4\delta f(f)}\Big)\,\theta\Big) \\
&= A^W \underbrace{-(n+2)\sum_\alpha \Big(df(Z_\alpha)\theta^\alpha - df(\overline{Z_\alpha})\theta^{\overline{\alpha}}\Big)}\ \underline{-4i(n+1)\delta f(f)\theta} \\
&\qquad -i\sum_\alpha \Big(\big(\nabla^W_{Z_\alpha} df\big)(\overline{Z_\alpha}) + \big(\nabla^W_{\overline{Z_\alpha}} df\big)(Z_\alpha)\Big)\,\theta.
\end{aligned}
$$

With the formula from Lemma 2.10,

$$
\hat{R}^W = e^{-2f}\Big(R^W - 2(n+1)\sum_k \Big(\big(\nabla^W_{Z_k} df\big)(\overline{Z_k}) + \big(\nabla^W_{\overline{Z_k}} df\big)(Z_k)\Big) - 4n(n+1)\delta f(f)\Big),
$$

we get for the connection $A^{\hat{\theta}}$:

$$
\begin{aligned}
A^{\hat{\theta}} &= \hat{A}^W - \tfrac{i}{2(n+1)}\hat{R}^W \cdot \hat{\theta} \\
&= A^W - (n+2)\sum_\alpha \Big(df(Z_\alpha)\theta^\alpha - df(\overline{Z_\alpha})\theta^{\overline{\alpha}}\Big) - 4i(n+1)\delta f(f)\theta \\
&\quad -i\sum_\alpha \Big(\big(\nabla^W_{Z_\alpha} df\big)(\overline{Z_\alpha}) + \big(\nabla^W_{\overline{Z_\alpha}} df\big)(Z_\alpha)\Big)\,\theta \\
&\quad -\tfrac{i}{2(n+1)}\Big(R^W - 2(n+1)\sum_k \Big(\big(\nabla^W_{Z_k} df\big)(\overline{Z_k}) + \big(\nabla^W_{\overline{Z_k}} df\big)(Z_k)\Big) \\
&\qquad\qquad -4n(n+1)\delta f(f)\Big)\cdot\theta \\
&= A^W - (n+2)\sum_\alpha \Big(df(Z_\alpha)\theta^\alpha - df(\overline{Z_\alpha})\theta^{\overline{\alpha}}\Big) - 2i(n+2)\delta f(f)\theta \\
&\quad -\tfrac{i}{2(n+1)}R^W\cdot\theta \\
&= A^\theta - (n+2)\sum_\alpha \Big(df(Z_\alpha)\theta^\alpha - df(\overline{Z_\alpha})\theta^{\overline{\alpha}}\Big) - 2i(n+2)\delta f(f)\theta.
\end{aligned}
$$

Now we can finally calculate the conformal factor between h_θ and $h_{\hat{\theta}}$. With the results above the metric $h_{\hat{\theta}}$ can be written as

$$
\begin{aligned}
h_{\hat{\theta}} &= \pi^* L_{\hat{\theta}} - i\tfrac{4}{n+2}\pi^*\hat{\theta}\odot A^{\hat{\theta}} \\
&= \pi^* e^{2f}\Big(L_\theta + \sum_\alpha \Big(\underbrace{2i\big(df(\overline{Z_\alpha})\theta^{\overline{\alpha}} - df(Z_\alpha)\theta^\alpha\big)\otimes\overline{\theta} - 2i\theta\otimes\big(df(Z_\alpha)\overline{\theta^{\overline{\alpha}}} - df(\overline{Z_\alpha})\overline{\theta^\alpha}\big)}_{=4i\theta\odot\left(df(\overline{Z_\alpha})\theta^{\overline{\alpha}} - df(Z_\alpha)\theta^\alpha\right)\ \text{in the real tangent space}}\Big) \\
&\qquad +8\delta f(f)\theta\otimes\overline{\theta}\Big) \\
&\quad -i\tfrac{4}{n+2}\pi^*\big(e^{2f}\theta\big)\odot\Big(A^\theta - (n+2)\sum_\alpha \big(df(Z_\alpha)\theta^\alpha - df(\overline{Z_\alpha})\theta^{\overline{\alpha}}\big) - 2i(n+2)\delta f(f)\theta\Big)
\end{aligned}
$$

and most of the terms cancel out

$$
\begin{aligned}
h_{\hat{\theta}} &= e^{2f\circ\pi}\Big(\pi^* L_\theta - i\tfrac{4}{n+2}\pi^*\theta\odot A^\theta\Big) \\
&\quad \underbrace{+4ie^{2f\circ\pi}\sum_\alpha \theta\odot\big(df(\overline{Z_\alpha})\theta^{\overline{\alpha}} - df(Z_\alpha)\theta^\alpha\big) + 8e^{2f\circ\pi}\delta f(f)\,\underbrace{\theta\otimes\overline{\theta}}_{=\theta\otimes\theta}}_{\text{cancel out}} \\
&\quad \overbrace{+4ie^{2f\circ\pi}\pi^*\theta\odot\sum_\alpha \big(df(Z_\alpha)\theta^\alpha - df(\overline{Z_\alpha})\theta^{\overline{\alpha}}\big) - 8e^{2f\circ\pi}\delta f(f)\theta\odot\theta} \\
&= e^{2f\circ\pi}h_\theta.
\end{aligned}
$$

Thus the conformal factor between both metrics is $e^{2f\circ\pi}$ and so the conformal class defined by the metric h_θ does not depend on the pseudo-hermitian form chosen, $[h_\theta] = [h_{\hat{\theta}}]$ as wanted.

$\square$

Definition 2.13 $(F, [h_\theta])$ *is called the Fefferman space of the strictly pseudo-convex CR manifold* (M^{2n+1}, T_{10}).

Later on we will explain another way ([CG08]) of defining a Fefferman space of a strictly pseudo-convex CR manifold. For this construction the tools of parabolic geometry are essential. Thus the next chapter will deal with Cartan and parabolic geometry.

Chapter 3

Cartan Geometry and Parabolic Geometry

In the language of Cartan geometry manifolds equipped with specific structures are viewed as "curved analogs" of homogeneous spaces. This idea was first introduced by Cartan in [Car52]. This setting is a generalization of the (pseudo-) Riemannian geometry as well as the Kleinian geometry. The Klein geometry established in 1872 by Klein in "Das Erlanger Programm" [Kl1872] considers manifolds (or just sets) with effective transformation groups and invariants of the given transformation group. Thus Klein geometry covers hyperbolical and elliptical geometries for example. In 1854 Riemann developed in his Habilitationsschrift the Riemannian geometry as curved analogs of the euclidian geometry. The Cartan geometry now covers Riemannian and Kleinian geometry. One very interesting aspect of Cartan geometry is that it allows to construct a boundary for the manifold out of intrinsic data as we will see later.

$$\begin{tikzcd}
 & \text{Cartan geometry} & \\
\text{Riemannian geometry} \arrow[ur] & & \text{Klein geometry} \arrow[ul] \\
\text{Euclidian geometry} \arrow[u] & & \text{elliptic, hyperbolic, projectove, } \ldots \text{ geometry} \arrow[u]
\end{tikzcd}$$

Given a Lie group G with Lie algebra $LA(G) = \mathfrak{g}$ and a closed subgroup $P \subset G$ a Cartan geometry of type (G, P) is a P-principal bundle $\pi : \mathcal{G} \longrightarrow M$ endowed with a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ which is a P-equivariant one form on $\mathcal{G}$ with values in the Lie algebra $\mathfrak{g}$ such that the generators of the fundamental vector fields are reproduced and the tangent bundle $T\mathcal{G}$ is trivialized by ω. A special class of Cartan geometries are the parabolic geometries. Here the Lie algebra $\mathfrak{g} = LA(G)$ is equipped with a $|k|$-grading. The name parabolic geometry is inspired by the fact that in the complex case those geometries are the ones with a parabolic subgroup P.

In this chapter the Cartan geometry and the parabolic geometry will be introduced and studied in detail, starting of with facts about Lie algebras and $|k|$-graded Lie algebras in particular in order to establish the denotations used further on.

3.1 Lie Algebras

Definition 3.1 *A Lie algebra* $\mathfrak{g}$ *over a field* $\mathbb{K}$ *is a* $\mathbb{K}$*-vector space endowed with a bilinear map* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ *satisfying*

- $[X, X] = 0$ *for all* $X \in \mathfrak{g}$*, that is to say* $[\cdot,\cdot]$ *is skew symmetric, and*
- *the Jacobi identity holds, i.e.* $\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big]$ *vanishes for all* $X, Y, Z \in \mathfrak{g}$.

The bilinear map $[\cdot,\cdot]$ *is called Lie bracket.*

Definition 3.2 *Let $\mathfrak{g}$ be a Lie algebra.*

- *A vector subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a subalgebra of $\mathfrak{g}$ if $\mathfrak{h}$ is closed under the Lie bracket, that is the Lie bracket maps $\mathfrak{h} \times \mathfrak{h}$ into $\mathfrak{h}$. Hence $(\mathfrak{h}, [\cdot,\cdot])$ is a Lie algebra itself.*
- *A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is an ideal of $\mathfrak{g}$ if $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$ holds.*
- *The center $\mathfrak{z}$ of a Lie algebra is the ideal defined by*

$$\mathfrak{z} := \{X \in \mathfrak{g} \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}.$$

- *The lower central series of a Lie algebra $\mathfrak{g}$ is defined by*

$$\underbrace{\mathcal{D}_0\mathfrak{g}}_{:=\mathfrak{g}} \supset \underbrace{\mathcal{D}_1\mathfrak{g}}_{:=[\mathfrak{g},\mathfrak{g}]} \supset \cdots \supset \underbrace{\mathcal{D}_k\mathfrak{g}}_{:=[\mathfrak{g},\mathcal{D}_{k-1}\mathfrak{g}]} \supset \cdots .$$

- *The derived series of a Lie algebra $\mathfrak{g}$ is defined by*

$$\underbrace{\mathcal{D}^0\mathfrak{g}}_{:=\mathfrak{g}} \supset \underbrace{\mathcal{D}^1\mathfrak{g}}_{:=[\mathfrak{g},\mathfrak{g}]} \supset \cdots \supset \underbrace{\mathcal{D}^k\mathfrak{g}}_{:=[\mathcal{D}^{k-1}\mathfrak{g},\mathcal{D}^{k-1}\mathfrak{g}]} \supset \cdots .$$

- *The adjoint representation of a Lie algebra $\mathfrak{g}$ is given by*

$$\begin{aligned} ad : \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g}) \\ X &\mapsto ad(X) \text{ with } ad(X)Y := [X, Y]. \end{aligned}$$

Definition 3.3 *A Lie algebra $\mathfrak{g}$ is called*

- *nilpotent, if $\mathcal{D}_k\mathfrak{g} = 0$ for some $k \in \mathbb{N}$,*
- *solvable, if $\mathcal{D}^k\mathfrak{g} = 0$ for some $k \in \mathbb{N}$,*
- *simple, if it is nonabelian and its only ideals are $\{0\}$ and $\mathfrak{g}$ itself, or*
- *semisimple, if $\mathfrak{g}$ has no nonzero solvable ideals.*

Definition 3.4 *The Killing form of a Lie algebra $\mathfrak{g}$ over the field $\mathbb{K}$ is defined by*

$$\begin{aligned} B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathbb{K} \\ X, Y &\mapsto Tr\big(ad(X) \circ ad(Y)\big). \end{aligned}$$

Definition 3.5 *Let G be a Lie group and $\mathfrak{g} = LA(G)$ its Lie algebra. The Adjoint action of G on $\mathfrak{g}$ is defined for $g \in G$ by*

$$\begin{aligned} Ad(g) : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ X &\mapsto Ad(g)X := dL_g \circ dR_{g^{-1}}X. \end{aligned}$$

Definition 3.6 *A derivation of a Lie algebra $\mathfrak{g}$ is a linear map $\varphi : \mathfrak{g} \longrightarrow \mathfrak{g}$ with*

$$\varphi\big([X, Y]\big) = \big[\varphi(X), Y\big] + \big[X, \varphi(Y)\big] \text{ for all } X, Y \in \mathfrak{g}.$$

Derivations of the type $ad(X) : \mathfrak{g} \longrightarrow \mathfrak{g}$ for $X \in \mathfrak{g}$ are called inner derivations.

The following lemma gives an overview about some commonly known facts on Lie algebras.

Lemma 3.1 *Let $\mathfrak{g}$ be a Lie algebra.*

- *The Killing form $B_{\mathfrak{g}}$ is invariant under all automorphisms of $\mathfrak{g}$. It is especially invariant under the Adjoint action of the Lie group G ($LA(G) = \mathfrak{g}$). I.e. for all $X, Y \in \mathfrak{g}$ and all $g \in G$ we have $B_{\mathfrak{g}}(Ad(g)X, Ad(g)Y) = B_{\mathfrak{g}}(X, Y)$.*

- *The Killing form $B_{\mathfrak{g}}$ is ad-invariant in the following way*

$$\text{for all } X, Y, Z \in \mathfrak{g} \text{ we have } B_{\mathfrak{g}}(ad(X)Y, Z) = -B_{\mathfrak{g}}(Y, ad(X)Z).$$

- *The Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form $B_{\mathfrak{g}}$ is nondegenerate.*

- *A semisimple Lie algebra $\mathfrak{g}$ is a direct sum of simple ideals.*

- *If $\mathfrak{g}$ is semisimple all derivations are inner derivations.*

3.2 $|k|$-graded Lie Algebras

The parabolic geometry profits considerably from the properties of the $|k|$-grading of the underlying Lie subalgebra $\mathfrak{p}$. So these properties need to be studied, which we do based on [CS00] and [CS03].

Definition 3.7 *Let $\mathbb{K}$ be the field of real or complex numbers, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ with a splitting $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ holds for all $i, j = -k, \ldots, k$ is called a $|k|$-graded Lie algebra over $\mathbb{K}$. Then $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ is called a $|k|$-grading of $\mathfrak{g}$.*

Definition 3.8 *Let $\mathfrak{g}$ be a Lie algebra with a $|k|$-grading. If furthermore the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$ and no simple ideal of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$ we call $\mathfrak{g}$ an effective semisimple $|k|$-graded Lie algebra.*

In the following we assume that $\mathfrak{g}$ is an effective semisimple $|k|$-graded Lie algebra. We denote by $\mathfrak{p}_+$ the subalgebra $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ of $\mathfrak{g}$ and with $\mathfrak{p}$ the subalgebra $\mathfrak{g}_0 \oplus \mathfrak{p}_+$.

Proposition 3.1 *Let $\mathfrak{g}$ be an effective semisimple $|k|$-graded Lie algebra.*

1. *There exists a uniquely defined element $E \in \mathfrak{g}_0$ with $[E, X] = lX$ for all $X \in \mathfrak{g}_l$. E is called the grading element. We have especially $[\mathfrak{g}_0, \mathfrak{g}_l] = \mathfrak{g}_l$.*

2. *For the Killing form $B_{\mathfrak{g}}$ we have $B_{\mathfrak{g}}(\mathfrak{g}_l, \mathfrak{g}_j) = 0$ for all l, j with $l + j \neq 0$. Furthermore for all $j = 1, \ldots, k$ the Killing form $B_{\mathfrak{g}}$ induces an isomorphism of the $\mathfrak{g}_0$-moduls $\mathfrak{g}_j^* \simeq \mathfrak{g}_{-j}$.*

3. *If $\mathfrak{g}'$ is an ideal in $\mathfrak{g}$, $\mathfrak{g}'$ is homogenous, that is to say $\mathfrak{g}' = \oplus_{i=-k}^{k}(\mathfrak{g}' \cap \mathfrak{g}_i)$. Especially $\mathfrak{g}$ is the direct sum of simple $|k_i|$-graded Lie algebras, where all k_i are smaller or equal to k and none is zero.*

4. *Let $A \in \mathfrak{g}_i$ with $i > -k$ be an element with $[A, X] = 0$ for all $X \in \mathfrak{g}_{-1}$. Then we have $A = 0$.*

5. *For $i < k$ it holds $[\mathfrak{g}_{i+1}, \mathfrak{g}_{-1}] = \mathfrak{g}_i$.*

Proof:

1. In order to find the grading element E we define the following derivation.

$$\begin{array}{rcl} D : \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ X & \mapsto & D(X) := j \cdot X \quad \text{for } X \in \mathfrak{g}_j \end{array}$$

   This is actually a derivation since for $X \in \mathfrak{g}_i$ and $Y \in \mathfrak{g}_j$ we have

$$\begin{aligned} D([X,Y]) &= (i+j)[X,Y] \\ &= [iX,Y] + [X,jY] \\ &= [D(X),Y] + [X,D(Y)]. \end{aligned}$$

   Now with $\mathfrak{g}$ being semisimple we know that all derivations are inner derivations, i.e. we have a unique element $E \in \mathfrak{g}$ with $D = ad(E)$. According to the grading of $\mathfrak{g}$ we can write $E = E_{-k} + \cdots + E_k$. We get

$$\begin{aligned} 0 &= [E,E] \\ &= \textstyle\sum_{j=-k}^{k} [E, E_j] \\ &= \textstyle\sum_{j=-k}^{k} jE_j. \end{aligned}$$

   Hence we obtain $E = E_0 \in \mathfrak{g}_0$.

   According to the grading we have $[\mathfrak{g}_0, \mathfrak{g}_l] \subset \mathfrak{g}_l$. With $[E,X] = lX$ for all $X \in \mathfrak{g}_l$ it actually has to be $[\mathfrak{g}_0, \mathfrak{g}_l] = \mathfrak{g}_l$.

2. We will now take a look at the properties of the Killing form with respect to the grading. Using that the Killing form is ad-invariant we have for $X \in \mathfrak{g}_l$ and $Y \in \mathfrak{g}_j$

$$\begin{aligned} lB_{\mathfrak{g}}(X,Y) &= B_{\mathfrak{g}}(ad(E)X, Y) \\ &= -B_{\mathfrak{g}}(X, ad(E)Y) \\ &= -jB_{\mathfrak{g}}(X,Y). \end{aligned}$$

   So for $l \neq -j$ we have $B_{\mathfrak{g}}(X,Y) = 0$. Since the Killing form is nondegenerate according to Lemma 3.1 the restriction $B_{\mathfrak{g}}|_{\mathfrak{g}_j \times \mathfrak{g}_{-j}}$ has to be nondegenerate as well. Hence

$$\begin{array}{rcl} B_{\mathfrak{g}} : \mathfrak{g}_j & \longrightarrow & \mathfrak{g}^*_{-j} \\ X & \mapsto & B_{\mathfrak{g}}(X, \cdot) \end{array}$$

   is an isomorphism which is actually an isomorphism of the $\mathfrak{g}_0$-modules due to the ad-invariance of the Killing form.

3. Let $\mathfrak{g}' \subset \mathfrak{g}$ be an ideal. An element $X \in \mathfrak{g}'$ can be written as a linear combination of vectors of $\mathfrak{g}_{-k}, \ldots, \mathfrak{g}_k$. Assume first that $X = X_i \oplus X_j$ with $X_i \in \mathfrak{g}_i$, $X_j \in \mathfrak{g}_j$, $i \neq j$. Using the grading element E and the fact that $\mathfrak{g}'$ is an ideal we obtain that also $[E, X_i \oplus X_j] = iX_i \oplus jX_j$ is an element of $\mathfrak{g}'$. Consequently also the linear combination $i \cdot (X_i \oplus X_j) - (iX_i \oplus jX_j) = (i-j)X_j$ is contained in the ideal $\mathfrak{g}'$. Inductively it follows that for $\oplus_j X_j \in \mathfrak{g}'$ each vector X_j with $X_j \neq 0$ is an element of $\mathfrak{g}'$. Thus we have the decomposition

$$\mathfrak{g}' = \bigoplus_{i=-k}^{k} (\mathfrak{g}' \cap \mathfrak{g}_i).$$

   With $\mathfrak{g}$ being semisimple it is the direct sum of simple ideals, i.e. the Lie algebra $\mathfrak{g}$ is the direct sum of simple $|k_i|$-graded Lie algebras, where all k_i are smaller or equal to k and none is zero, since for an effective semisimple graded Lie algebra $\mathfrak{g}$ we require that no simple ideal is contained in $\mathfrak{g}_0$.

4. Next we prove that the vanishing of $[A, X]$ for some $A \in \mathfrak{g}_i$, $i > -k$, and all $X \in \mathfrak{g}_{-1}$ implies $A = 0$. Since $\mathfrak{g}_{-1}$ generates $\mathfrak{g}_-$ we have $[\mathfrak{g}_{i+1}, \mathfrak{g}_{-1}] = \mathfrak{g}_i$ for all $i < -1$ and using the duality we also have $[\mathfrak{g}_{i-1}, \mathfrak{g}_1] = \mathfrak{g}_i$ for all $i > 1$. Consequently a linear subspace $\mathfrak{a} \subset \mathfrak{g}$ is an ideal if $[\mathfrak{g}_i, \mathfrak{a}] \subset \mathfrak{a}$ for $i = -1, 0, 1$.

   We define for integers l, $-k \leq l \leq k$
   $$\mathfrak{a}^l(l) := \{X \in \mathfrak{g}_l \mid [X, \mathfrak{g}_{-1}] = 0\}.$$
   Further we set $\mathfrak{a}^l(m+1) := [\mathfrak{a}^l(m), \mathfrak{g}_1] \subset \mathfrak{g}_{m+1}$ and $\mathfrak{a}^l := \bigoplus_{m=l}^k \mathfrak{a}^l(m)$. We want to prove that $\mathfrak{a}^l$ is an ideal.

   According to the definition we have $[\mathfrak{a}^l, \mathfrak{g}_1] \subset \mathfrak{a}^l$.

   For all $X \in \mathfrak{a}^l(l)$, $X_0 \in \mathfrak{g}_0$ and $X_{-1} \in \mathfrak{g}_{-1}$ we obtain, using the Jacobi identity,
   $$\big[[X, X_0], X_{-1}\big] = -\big[\underbrace{[X_0, X_{-1}]}_{\in \mathfrak{g}_{-1}}, X\big] - \big[\underbrace{[X_{-1}, X]}_{=0}, X_0\big] = 0.$$
   That means $[\mathfrak{a}^l(l), \mathfrak{g}_0] \subset \mathfrak{a}^l(l)$. Assume we have $[\mathfrak{a}^l(m), \mathfrak{g}_0] \subset \mathfrak{a}^l(m)$ then we also have $[\mathfrak{a}^l(m+1), \mathfrak{g}_0] \subset \mathfrak{a}^l(m+1)$ since all elements of $\mathfrak{a}^l(m+1)$ are of the shape $[X, X_1]$ with $X \in \mathfrak{a}^l(m)$ and $X_1 \in \mathfrak{g}_1$ and we have for any $X_0 \in \mathfrak{g}_0$
   $$\big[[X, X_1], X_0\big] \overset{\text{Jac.}}{=} -\big[\underbrace{[X_1, X_0]}_{\in \mathfrak{g}_1}, \underbrace{X}_{\in \mathfrak{a}^l(m)}\big] - \big[\underbrace{[X_0, X]}_{\in \mathfrak{a}^l(m)}, X_1\big] \in \mathfrak{a}^l(m+1).$$
   Thus we obtain $[\mathfrak{a}^l(m), \mathfrak{g}_0] \subset \mathfrak{a}^l(m)$ for $m = l, \ldots, k$ and therefore $[\mathfrak{a}^l, \mathfrak{g}_0] \subset \mathfrak{a}^l$. Finally we prove that $[\mathfrak{a}^l, \mathfrak{g}_{-1}] \subset \mathfrak{a}^l$, then we know that $\mathfrak{a}^l$ is an ideal. According to the definition we have $[\mathfrak{a}^l(l), \mathfrak{g}_{-1}] = 0 =: \mathfrak{a}^l(l-1)$. Assuming that $[\mathfrak{a}^l(m), \mathfrak{g}_{-1}] \subset \mathfrak{a}^l(m-1)$ is true, we get $[\mathfrak{a}^l(m+1), \mathfrak{g}_{-1}] \subset \mathfrak{a}^l(m)$, since for all $[X, X_1] \in \mathfrak{a}^l(m+1)$ with $X \in \mathfrak{a}^l(m)$ and $X_1 \in \mathfrak{g}_1$ and all $X_{-1} \in \mathfrak{g}_{-1}$ we have
   $$\big[[X, X_1], X_{-1}\big] \overset{\text{Jac.}}{=} -\big[\underbrace{[X_1, X_{-1}]}_{\in \mathfrak{g}_0}, \underbrace{X}_{\in \mathfrak{a}^l(m)}\big] - \big[\underbrace{[X_{-1}, X]}_{\in \mathfrak{a}^l(m-1)}, X_1\big] \in \mathfrak{a}^l(m).$$
   Consequently $[\mathfrak{a}^l, \mathfrak{g}_{-1}] \subset \mathfrak{a}^l$ and $\mathfrak{a}^l$ is an ideal. For $l > -k$ the ideal $\mathfrak{a}^l$ is not the whole Lie algebra, $\mathfrak{a}^l \neq \mathfrak{g}$. If the Lie algebra $\mathfrak{g}$ is simple $\mathfrak{a}^l = \{0\}$ follows. As we have just seen in the semisimple case $\mathfrak{g} = \mathfrak{b}_1 \oplus \cdots \mathfrak{b}_r$ is a direct sum of simple ideals $\mathfrak{b}_j$ endowed with a $|k_j|$-grading and since no simple ideal of $\mathfrak{g}$ is supposed to be contained in $\mathfrak{g}_0$ each k_j is nonzero. Thus we can apply the construction above on each $\mathfrak{b}_j$ and obtain the result wanted, $\mathfrak{a}^l = \{0\}$ and so $[A, X] = 0$ for some $A \in \mathfrak{g}_i$, $i > -k$, and all $X \in \mathfrak{g}_{-1}$ implies $A = 0$.

5. Finally we will see that $[\mathfrak{g}_{i+1}, \mathfrak{g}_{-1}] = \mathfrak{g}_i$ holds for all $i < k$. As just discussed $\mathfrak{a}^{-k}$ is an ideal and with $[\mathfrak{g}_{-k}, \mathfrak{g}_{-1}] = 0$ we get
   $$\begin{aligned} \mathfrak{a}^{-k} &= \mathfrak{a}^{-k}(-k) \oplus [\mathfrak{a}^{-k}(-k), \mathfrak{g}_1] \oplus \cdots \\ &= \mathfrak{g}_{-k} \oplus [\mathfrak{g}_{-k}, \mathfrak{g}_1] \oplus \cdots . \end{aligned}$$
   As above we apply the construction to the simple ideals of $\mathfrak{g} = \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_r$ and obtain that each $\mathfrak{a}^{-k}_{\mathfrak{b}_j}$ is nontrivial and therefore the whole ideal. Consequently successively applying $ad(\mathfrak{g}_1)$ to $\mathfrak{g}_{-k}$ generates the whole Lie algebra and we obtain
   $$[\mathfrak{g}_{i-1}, \mathfrak{g}_1] = \mathfrak{g}_i \text{ for } i > -k.$$

Dualization gives the result wanted

$$[\mathfrak{g}_{i+1}, \mathfrak{g}_{-1}] = \mathfrak{g}_i \text{ for } i < k.$$

□

The subalgebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k \subset \mathfrak{g}$ is a parabolic subalgebra of $\mathfrak{g}$, that is it contains a Borel subalgebra (a maximal solvable subalgebra of $\mathfrak{g}$).
With the help of the Killing form $B_\mathfrak{g}$ the grading of $\mathfrak{g}$ is completely determined by $\mathfrak{p}$, because with $\mathfrak{p}$ we also have $\mathfrak{p}^* = \mathfrak{g}_- \oplus \mathfrak{g}_0$ and $\mathfrak{g}_0 = \mathfrak{p} \cap \mathfrak{p}^*$. Then $\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is given as the orthogonal complement of $\mathfrak{g}_0$ in $\mathfrak{p}^*$ with respect to the Killing form. Since $\mathfrak{g}_{-1}$ generates $\mathfrak{g}_-$ we have the following splitting of $\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ into the orthogonal complements

$$\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-2} = [\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}, \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}] \text{ and } \mathfrak{g}_{-1}$$

and so on.
With $\mathfrak{g}^i := \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$ we obtain a filtration $\{0\} \subset \mathfrak{g}^k \subset \mathfrak{g}^{k-1} \subset \cdots \subset \mathfrak{g}^{-k} = \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$.
We now define

$$P := \{g \in G \mid Ad(g)(\underbrace{\mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_k}_{=:\mathfrak{g}^j}) \subset \mathfrak{g}^j \text{ for all } j = -k, \ldots, k\}.$$

P is a closed subgroup of G. For $X \in LA(P)$ we have a curve $\gamma : (-\varepsilon, \varepsilon) \longrightarrow P$ representing $X = \dot{\gamma}(0)$. Hence we have $Ad(\gamma(t))(\mathfrak{g}^i) \subset \mathfrak{g}^i$ and therefore $ad(X)(\mathfrak{g}^i) \subset \mathfrak{g}^i$. So the Lie algebra of P is contained in $\mathfrak{p}$. Let the other way round X be an element of $\mathfrak{p}$. We want to show, that $\exp(tX)$ is an element of P.

Lemma 3.2 *For every $X_0 \in \mathfrak{g}_0$ the map $Ad(\exp(tX_0))$ preserves the grading of the Lie algebra $\mathfrak{g}$, $Ad(\exp(tX_0)) : \mathfrak{g}_i \longrightarrow \mathfrak{g}_i$, $i = -k, \ldots, k$. For every $X_l \in \mathfrak{g}_l$, $l > 0$ the map $Ad(\exp(tX_l))$ preserves the filtration of the Lie algebra $\mathfrak{g}$, $Ad(\exp(tX_l)) : \mathfrak{g}_i \longrightarrow \mathfrak{g}^i$ with $i = -k, \ldots, k$. Especially we have $\exp(tX) \in P$ for all $X \in \mathfrak{p}$.*

Proof: Let $X_l \in \mathfrak{g}_l$ with $l \geq 0$. Using the grading element E we have

$$\begin{aligned} \tfrac{d}{dt}\big(Ad(\exp(tX_l))(E)\big)\big|_{t=s} &= Ad(\exp(sX_l))\,\underbrace{[X_l, E]}_{=-lX_l} \\ &= -lX_l \\ &= const. \end{aligned}$$

Thus we get $Ad(\exp(tX_l))(E) = -ltX_l + const$, where the constant part is given by the value for $t = 0$, $const = Ad(exp(0))(E) = E$. We obtain

$$Ad(\exp(tX_l))(E) = E - ltX_l.$$

Let further $Y \in \mathfrak{g}_i$. We have

$$\begin{aligned} i \cdot Ad(\exp(tX_l))(Y) &= Ad(exp(tX_l))[E, Y] \\ &= \big[Ad(\exp(tX_l))(E), Ad(\exp(tX_l))(Y)\big] \\ &= \big[E - ltX_l, Ad(exp(tX_l))(Y)\big]. \end{aligned}$$

We denote the projections of $Ad(\exp(tX_l))(Y)$ to $\mathfrak{g}_j$ by $\widetilde{Y}_j$ and can write

$$\begin{aligned} i(\widetilde{Y}_{-k} \oplus \cdots \oplus \widetilde{Y}_k) &= \big[E - ltX_l, \widetilde{Y}_{-k} \oplus \cdots \oplus \widetilde{Y}_k\big] \\ &= -k\widetilde{Y}_{-k} \oplus \cdots \oplus k\widetilde{Y}_k - lt\big[X_l, \widetilde{Y}_{-k} \oplus \cdots \oplus \widetilde{Y}_k\big]. \end{aligned}$$

This is equivalent to

$$0 = (i-j)\widetilde{Y}_j + lt[X_l, \widetilde{Y}_{j-l}] \text{ for } j = -k, \ldots, k.$$

First case $l = 0$

We have $0 = (i-j)\widetilde{Y}_j$ for $j = -k, \ldots, k$. Thus we obtain $\widetilde{Y}_j = 0$ for all $j \neq i$ and therefore

$$\exp(tX_0) : \mathfrak{g}_i \longrightarrow \mathfrak{g}_i \text{ for all } X_0 \in \mathfrak{g}_0 \text{ and } i = -k, \ldots, k.$$

Second case $l > 0$

Assume that $\widetilde{Y}_{-k}, \ldots, \widetilde{Y}_\eta = 0$ (this is trivially true for $\eta < -k$). If $\eta + 1$ is smaller than i than $\widetilde{Y}_{\eta+1}$ vanishes as well since

$$0 = \underbrace{(i - (\eta+1))}_{\neq 0} \widetilde{Y}_{\eta+1} + lt[X_l, \underbrace{\widetilde{Y}_{\eta+1-l}}_{=0}].$$

Consequently

$$\exp(tX_l) : \mathfrak{g}_i \longrightarrow \mathfrak{g}^i \text{ for all } X_l \in \mathfrak{g}_l,\ l > 0 \text{ and } i = -k, \ldots, k.$$

Especially we have $\exp(tX) \in P$ for all $X \in \mathfrak{p}$, i.e. $\mathfrak{p} \subset LA(P)$.

□

We can conclude that the Lie algebra of P is $\mathfrak{p}$. We set

$$G_0 := \{g \in G \mid Ad(g)\mathfrak{g}_i \subset \mathfrak{g}_i \text{ for all } i = -k, \ldots, k\}.$$

G_0 is a subgroup of P, actually the reductive part of P and analogously to $LA(P) = \mathfrak{p}$ it holds $LA(G_0) = \mathfrak{g}_0$.

Lemma 3.3 *For $l \geq 1$, $X_l \in \mathfrak{g}_l$ and $Y \in \mathfrak{g}_i$ we have*

$$pr_{\mathfrak{g}^i/_{\mathfrak{g}^{i+l}}} \circ Ad\big(\exp(X_l)\big)(Y) = Y.$$

Proof: Let X_l be an element of $\mathfrak{g}_l$ with $l \geq 1$ and $Y \in \mathfrak{g}_i$. Thus $\exp(tX_l)$ is an element of P for any $t \in \mathbb{R}$ and $Ad(\exp(tX_l))$ respects the filtration of the Lie algebra $\mathfrak{g}$ as we have seen in Lemma 3.2. Then we can calculate for any $s \in \mathbb{R}$

$$\begin{aligned} \tfrac{d}{dt}\big(Ad\big(\exp(tX_l)\big)(Y)\big)\Big|_{t=s} &= \tfrac{d}{dt}\big(Ad\big(\exp(sX_l)\big) \circ Ad\big(\exp(tX_l)\big)(Y)\big)\Big|_{t=0} \\ &= Ad\big(\exp(sX_l)\big) \circ \underbrace{ad(X_l)(Y)}_{\in \mathfrak{g}_{i+l}} \\ &\in \mathfrak{g}^{i+l}. \end{aligned}$$

Consequently the $\mathfrak{g}^i/_{\mathfrak{g}^{i+l}}$-component of $Ad\big(\exp(tX_l)\big)(Y)$ is constant and thus equal to $Ad\big(\exp(0 \cdot X_l)\big)(Y) = Y$.

□

Lemma 3.4 *For every element $p \in P$ there are unique elements $g_0 \in G_0$ and $X_i \in \mathfrak{g}_i$, for $i = 1, \ldots, k$, such that $p = g_0 \exp(X_1) \cdots \exp(X_k)$.*

Proof: Given an element $p \in P$ the Adjoint action $Ad(p) : \mathfrak{g} \longrightarrow \mathfrak{g}$ respects the filtration and more precisely $\mathfrak{g}_i$ is mapped to $\mathfrak{g}^i = \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$. We define the map $\varphi_0 : \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$\begin{aligned} \varphi_0(\underbrace{Y_{-k} \oplus \cdots \oplus Y_k}_{=Y}) &:= \textstyle\sum_{i=-k}^{k} pr_{\mathfrak{g}_i} \circ Ad(p) \circ pr_{\mathfrak{g}_i}(Y) \\ &= pr_{\mathfrak{g}_{-k}} \circ Ad(p)(Y_{-k}) \oplus \cdots \oplus pr_{\mathfrak{g}_k} \circ Ad(p)(Y_k). \end{aligned}$$

φ_0 is an automorphism of the graded Lie algebra $\mathfrak{g}$ and $\varphi_0^{-1} = \sum_{i=-k}^{k} pr_{\mathfrak{g}_i} \circ Ad(p^{-1}) \circ pr_{\mathfrak{g}_i}$. Furthermore for $Y \in \mathfrak{g}_i$ we have $\varphi_0(Y) = pr_{\mathfrak{g}_i} \circ Ad(p)(Y)$ which is congruent to $Ad(p)(Y)$ modulo $\mathfrak{g}^{i+1}$. We define $\varphi_1 := \varphi_0^{-1} \circ Ad(p)$ which satisfies especially

$$\begin{gathered} \varphi_1(E) \equiv E \; mod \; \mathfrak{g}^1 \\ \text{and for all } Y \in \mathfrak{g}_i \\ \varphi_1(Y) \equiv Y \; mod \; \mathfrak{g}^{i+1}. \end{gathered}$$

We set $X_1 := -pr_{\mathfrak{g}_1} \circ \varphi_0^{-1} \circ Ad(p)(E)$, that is to say $\varphi_0^{-1} \circ Ad(p)(E)$ is congruent to $E - X_1$ modulo $\mathfrak{g}^2$. As we have seen in the proof of Lemma 3.2 it holds $Ad(\exp(-X_1))(E) = E + X_1$ and further we have $Ad(\exp(-X_1))(X_1) = X_1$. Consequently we obtain

$$Ad\big(\exp(-X_1)\big)(E - X_1) = E.$$

Now we can define the automorphism $\varphi_2 := Ad\big(\exp(-X_1)\big) \circ \varphi_0^{-1} \circ Ad(p)$ satisfying

$$\begin{aligned} \varphi_2(E) &= Ad\big(\exp(-X_1)\big) \circ \underbrace{\varphi_0^{-1} \circ Ad(p)(E)}_{\equiv (E - X_1) \; mod \; \mathfrak{g}^2} \\ &\equiv E \; mod \; \mathfrak{g}^2 \end{aligned}$$

and for $Y \in \mathfrak{g}_i$ using Lemma 3.3

$$\begin{aligned} \varphi_2(Y) &= Ad\big(\exp(-X_1)\big) \circ \underbrace{\varphi_0^{-1} \circ Ad(p)(Y)}_{\equiv Y \; mod \; \mathfrak{g}^{i+1}} \\ &\equiv Ad\big(\exp(-X_1)\big)(Y) \; mod \; \mathfrak{g}^{i+1} \\ &\equiv Y \; mod \; \mathfrak{g}^{i+1}. \end{aligned}$$

Inductively we obtain elements $X_j \in \mathfrak{g}_j$ and automorphisms $\varphi_j = Ad\big(\exp(-X_{j-1})\big) \circ \varphi_{j-1}$ of the Lie algebra $\mathfrak{g}$ satisfying $\varphi_j(E) \equiv E \; mod \; \mathfrak{g}^j$ and $\varphi_j(Y) \equiv Y \; mod \; \mathfrak{g}^{i+1}$ for all $Y \in \mathfrak{g}_i$. So for φ_{k+1} we find $\varphi_{k+1}(E)$ is congruent to E modulo $\mathfrak{g}^{k+1} = \{0\}$, i.e. $\varphi_{k+1}(E) = E$. For $Y \in \mathfrak{g}_i$ the element $\varphi_{k+1}(Y)$ is congruent to Y modulo $\mathfrak{g}^{i+1}$. However with

$$\begin{aligned} [E, \varphi_{k+1}(Y)]_{\mathfrak{g}} &= [\varphi_{k+1}(E), \varphi_{k+1}(Y)]_{\mathfrak{g}} \\ &= \varphi_{k+1} \underbrace{[E, Y]_{\mathfrak{g}}}_{=iY} \\ &= i\varphi_{k+1}(Y) \end{aligned}$$

we find that $\varphi_{k+1}(Y)$ is actually an element of $\mathfrak{g}_i$ and thus $\varphi_{k+1}(Y) = Y$ for all $Y \in \mathfrak{g}_i$. Consequently $\varphi_{k+1} = Ad\big(\exp(-X_k)\big) \circ \cdots \circ Ad\big(\exp(-X_1)\big) \circ \varphi_0^{-1} \circ Ad(p)$ is the identity. Therefore, φ_0 is the Adjoint action of $\widetilde{g_0} := p \exp(-X_k) \cdots \exp(-X_1)$ which is an element of G_0 since φ_0 respects the grading of the Lie algebra $\mathfrak{g}$. With p and $\widetilde{g_0} \exp(X_1) \cdots \exp(X_k)$ having the same Adjoint action they only differ by an element g of the center of G which however is contained in G_0 since G is semisimple. I.e. we obtain a representation of $p \in P$ in the requested way

$$p = g_0 \exp(X_1) \cdots \exp(X_k) \text{ with } g_0 \in G_0 \text{ and } X_i \in \mathfrak{g}_i, \; i = 1, \ldots, k.$$

Assume that $g_0 \exp(X_1) \cdots \exp(X_k) = \hat{g}_0 \exp(\hat{X}_1) \cdots \exp(\hat{X}_k)$ is another representation of $p \in P$.
Then we can write for the Adjoint action of $\hat{g}_0$ on an element $Y \in \mathfrak{g}_i$

$$\begin{aligned} Ad(\hat{g}_0)(Y) &= Ad\big(g_0 \cdot \exp(X_1) \cdots \exp(X_k) \cdot \exp(-\hat{X}_k) \cdots \exp(-\hat{X}_1)\big)(Y) \\ &= Ad(g_0) \circ \underbrace{Ad\big(\exp(X_1) \cdots \exp(X_k) \cdot \exp(-\hat{X}_k) \cdots \exp(-\hat{X}_1)\big)(Y)}_{\equiv Y \ mod \ \mathfrak{g}^{i+1} \text{ according to Lemma 3.3}} \\ &\equiv Ad(g_0)(Y) \ mod \ \mathfrak{g}^{i+1}. \end{aligned}$$

However the Adjoint actions of $\hat{g}_0$ and g_0 respect the grading of the Lie algebra $\mathfrak{g}$. Thus we actually have $Ad(\hat{g}_0) = Ad(g_0)$ and consequently

$$Ad\big(\exp(X_1) \cdots \exp(X_k)\big) = Ad\big(\exp(\hat{X}_1) \cdots \exp(\hat{X}_k)\big).$$

Applying this to the grading element $E \in \mathfrak{g}_0$ using Lemma 3.3 again we obtain that $Ad(\exp(X_1))(E)$ is congruent to $Ad(\exp(\hat{X}_1))(E)$ modulo $\mathfrak{g}^2$. Recall the equation $Ad(\exp(tX_1))(E) = E - tX_1$. In the same way $Ad(\exp(t\hat{X}_1))(E) = E - t\hat{X}_1$ holds. Hence X_1 is actually equal to $\hat{X}_1$. Inductively we obtain $X_i = \hat{X}_i$ for all $i = 1, \ldots, k$. Thus we also have $g_0 = \hat{g}_0$. I.e. for every element $p \in P$ there are unique elements $g_0 \in G_0$ and $X_i \in \mathfrak{g}_i$, for $i = 1, \ldots, k$, such that $p = g_0 \exp(X_1) \cdots \exp(X_k)$.

□

We now define the subgroup $P_+ := \exp(\mathfrak{p}_+) \subset P$. And we furthermore obtain $P/_{P_+} \simeq G_0$, that is P is actually a semi direct product $P = G_0 \ltimes P_+$.
Note that we also have $[\mathfrak{g}_1, \mathfrak{g}_{j-1}] = \mathfrak{g}_j$ for $j > -k$. In Section 3.3 we will prove the existence of an involutive automorphism $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$ which is compatible with the Lie bracket and satisfies $\sigma(\mathfrak{g}_j) = \mathfrak{g}_{-j}$. With the help of σ we have for all $j > -k$

$$\begin{aligned} [\mathfrak{g}_1, \mathfrak{g}_{j-1}] &= \big[\sigma(\mathfrak{g}_{-1}), \sigma(\mathfrak{g}_{1-j})\big] \\ &= \sigma\big([\mathfrak{g}_{-1}), \mathfrak{g}_{1-j}]\big) \\ &= \sigma(\mathfrak{g}_{-j}) \\ &= \mathfrak{g}_j. \end{aligned}$$

This implies especially that the powers of $\mathfrak{p}_+$ are given by $\mathfrak{p}_+^i = \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$. Further $P_+^i := \exp(\mathfrak{p}_+^i) = \exp(\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k)$ and we set for $l \geq 0$ $G_l := P/_{P_+^{l+1}}$ extending the notation $G_0 = P/_{P_+} = P/_{P_+^1}$. Please note that P_+^{i+1} acts trivial on $\mathfrak{g}^j/_{\mathfrak{g}^{j+i}}$.
Later on we will need informations about the dual of the adjoint action of P restricted to $\mathfrak{g}_i$. For $p \in P$ and $X \in \mathfrak{g}_i$ we have $\big(pr_{\mathfrak{g}_i} \circ Ad(p)\big)^* X^* := \big(pr_{\mathfrak{g}_i} \circ Ad(p)X\big)^* \in \mathfrak{g}_{-i}$. With the help of the $Ad(P)$-invariant Killing form and keeping in mind that $B(\mathfrak{g}_k, \mathfrak{g}_l) = 0$ for $k + l \neq 0$ we can write

$$\begin{aligned} 1 &= B(X^*, X) \\ &= B\big(\underbrace{Ad(p)X^*}_{\in \mathfrak{g}^{-i}}, \underbrace{Ad(p)X}_{\in \mathfrak{g}^i}\big) \\ &= B\big(pr_{\mathfrak{g}_{-i}} \circ Ad(p)X^*, pr_{\mathfrak{g}_i} \circ Ad(p)X\big). \end{aligned}$$

So according to the definition of the dual we get $\big(pr_{\mathfrak{g}_i} \circ Ad(p)X\big)^* = pr_{\mathfrak{g}_{-i}} \circ Ad(p)X^*$ for all $p \in P$ and all $X \in \mathfrak{g}_i$ and we obtain the following lemma.

Lemma 3.5 *For all $i \in \{-k, \ldots, -1, 1, \ldots, k\}$ we have*

$$(pr_{\mathfrak{g}_i} \circ Ad)^* = pr_{\mathfrak{g}_{-i}} \circ Ad.$$

3.3 Cohomology Groups

In this section based on [CS00] we want to examine the cohomology groups of $\mathfrak{g}_-$ with coefficients in $\mathfrak{g}$. The cohomology groups of $|k|$-graded Lie algebras have to be studied, since parts of the first cohomology presents an obstacle in the prolongation procedure and the second cohomology is connected to possible values of the curvature of the normal Cartan connection.
The cochains of this cohomology are defined in the usual way to be the space of the linear maps from the nth exterior power of $\mathfrak{g}_-$ to the Lie algebra $\mathfrak{g}$: $C^n(\mathfrak{g}_-, \mathfrak{g}) := L(\bigwedge^n \mathfrak{g}_-, \mathfrak{g})$. It is also possible the see the cochains as multilinear skew symmetric maps. As is customary the differential $\partial : C^n(\mathfrak{g}_-, \mathfrak{g}) \longrightarrow C^{n+1}(\mathfrak{g}_-, \mathfrak{g})$ is given by

$$\begin{aligned}(\partial\varphi)(X_0, \dots, X_n) \;:=\; & \textstyle\sum_{i=0}^n (-1)^i \big[X_i, \varphi(X_0, \dots, \hat{X}_i, \dots, X_n)\big]_{\mathfrak{g}} \\ & + \textstyle\sum_{i<j} (-1)^{i+j} \varphi\big([X_i, X_j]_{\mathfrak{g}}, X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_n\big),\end{aligned}$$

where $\hat{X}_i$ means that X_i has to be omitted.
The subgroup P acts on the cochains $\psi : \bigwedge^n \mathfrak{g}_-^* \longrightarrow \mathfrak{g}$ via the Adjoint action, that is to say the cochain $Ad(p)\varphi$ maps $(Ad(p)X_1, \dots, Ad(p)X_n)$ to $Ad(p) \circ \varphi(X_1, \dots, X_n)$. And according to the definition of the differential $\partial(Ad(p)\varphi)$ maps $(Ad(p)X_0, \dots, Ad(p)X_n)$ to $Ad(p) \circ \partial\varphi(X_0, \dots, X_n)$. Thus the differential is P-equivariant, $\partial(Ad(p)\varphi) = Ad(p)(\partial\varphi)$.
The cohomology groups are defined as usual as the cocycles divided by the coboundaries,

$$H^n(\mathfrak{g}_-, \mathfrak{g}) := \frac{Ker\big(\partial : C^n(\mathfrak{g}_-, \mathfrak{g}) \longrightarrow C^{n+1}(\mathfrak{g}_-, \mathfrak{g})\big)}{Im\big(\partial : C^{n-1}(\mathfrak{g}_-, \mathfrak{g}) \longrightarrow C^n(\mathfrak{g}_-, \mathfrak{g})\big)}.$$

With $C^n_l(\mathfrak{g}_-, \mathfrak{g})$ we denote the space of the linear maps which are homogeneous of degree l, i.e. which satisfy $\varphi(X_1, \dots, X_n) \in \mathfrak{g}_{i_1+\dots+i_n+l}$ if $X_j \in \mathfrak{g}_{i_j}$ holds. Directly form the definition of the differential ∂ we can conclude that ∂ maps the space $C^n_l(\mathfrak{g}_-, \mathfrak{g})$ to $C^{n+1}_l(\mathfrak{g}_-, \mathfrak{g})$. Consequently the cohomology groups split with respect to the degree $H^n(\mathfrak{g}_-, \mathfrak{g}) = \bigoplus_l H^n_l(\mathfrak{g}_-, \mathfrak{g})$.
The subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ acts via the adjoint action on each of the components $\mathfrak{g}_i$. This implies an action on the spaces $C^n(\mathfrak{g}_-, \mathfrak{g})$, which preserves the homogeneity of the maps. Furthermore the differential ∂ is a homomorphism of the $\mathfrak{g}_0$-moduls. Consequently the cohomology groups $H^n_l(\mathfrak{g}_-, \mathfrak{g})$ are $\mathfrak{g}_0$-moduls as well.
As we have seen in the section above (3.2) we can identify the subalgebra $\mathfrak{g}_-$ with the dual of the subalgebra $\mathfrak{p}_+$ with the help of the Killing form $B_{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$. By fixing a basis (ξ_α) of $\mathfrak{g}_-$ and denoting the corresponding dual basis of $\mathfrak{p}_+$ by (η_α) we obtain the dual pairing of $C^n(\mathfrak{g}_-, \mathfrak{g})$ and $C^n(\mathfrak{p}_+, \mathfrak{g})$ via

$$\begin{aligned}\langle\varphi, \psi\rangle \;&=\; \textstyle\sum_{\alpha_1<\dots<\alpha_n} B_{\mathfrak{g}}\big(\varphi(\xi_{\alpha_1}, \dots, \xi_{\alpha_n}), \psi(\eta_{\alpha_1}, \dots, \eta_{\alpha_n})\big) \\ &=\; \tfrac{1}{n!} \textstyle\sum_{\alpha_1,\dots,\alpha_n} B_{\mathfrak{g}}\big(\varphi(\xi_{\alpha_1}, \dots, \xi_{\alpha_n}), \psi(\eta_{\alpha_1}, \dots, \eta_{\alpha_n})\big).\end{aligned}$$

Note that this definition is independent of the basis chosen.
We define the codifferential $\partial^* : C^{n+1}(\mathfrak{g}_-, \mathfrak{g}) \longrightarrow C^n(\mathfrak{g}_-, \mathfrak{g})$ to be the negative dual operator to $\partial : C^n(\mathfrak{p}_+, \mathfrak{g}) \longrightarrow C^{n+1}(\mathfrak{p}_+, \mathfrak{g})$, that is for all $\varphi \in C^{n+1}(\mathfrak{g}_-, \mathfrak{g})$ and all $\psi \in C^n(\mathfrak{p}_+, \mathfrak{g})$ we require

$$\langle\partial^*\varphi, \psi\rangle \stackrel{!}{=} -\langle\varphi, \partial\psi\rangle.$$

As the differential the codifferential satisfies $\partial^* \circ \partial^* = 0$.
The codifferential inherits the property of being P-equivariant directly from the differential and the Killing form.
Since the Killing form $B_{\mathfrak{g}}$ identifies the $\mathfrak{g}_0$-moduls $\mathfrak{g}_-$ and the dual of $\mathfrak{p}_+$, the codifferential ∂^* is as well a homomorphism of $\mathfrak{g}_0$-moduls.

Later on we will need an explicit formula of the codifferential ∂^* acting on bilinear maps, $\partial^* : C^2(\mathfrak{g}_-, \mathfrak{g}) \longrightarrow C^1(\mathfrak{g}_-, \mathfrak{g})$. According to the definition we have $\langle \partial^*\varphi, \psi\rangle = -\langle \varphi, \partial\psi\rangle$. For $\varphi \in C^2(\mathfrak{g}_-, \mathfrak{g})$ and $\psi \in C^1(\mathfrak{p}_+, \mathfrak{g})$ we get

$$\begin{aligned} -\langle \varphi, \partial\psi\rangle &= \tfrac{1}{2}\textstyle\sum_{\alpha,\beta} B_{\mathfrak{g}}\big(\varphi(\xi_\alpha, \xi_\beta), \partial\psi(\eta_\alpha, \eta_\beta)\big) \\ &= \tfrac{1}{2}\textstyle\sum_{\alpha,\beta} B_{\mathfrak{g}}\big(\varphi(\xi_\alpha, \xi_\beta), -[\eta_\alpha, \psi(\eta_\beta)] + [\eta_\beta, \psi(\eta_\alpha)] + \psi([\eta_\alpha, \eta_\beta])\big). \end{aligned}$$

With the bilinearity of the Killing form $B_{\mathfrak{g}}$ this splits into three terms, of which the first two each yield

$$\frac{1}{2}\sum_\beta B_{\mathfrak{g}}\left(\sum_\alpha \big[\eta_\alpha, \varphi(\xi_\alpha, \xi_\beta)\big], \psi(\eta_\beta)\right).$$

In order to calculate the third term we write $[\eta_\alpha, \eta_\beta] = \sum_\gamma a^\gamma_{\alpha\beta}\eta_\gamma$, where the coefficients are given by $a^\gamma_{\alpha\beta} := B_{\mathfrak{g}}\big(\xi_\gamma, [\eta_\alpha, \eta_\beta]\big)$. Using the *ad*-invariance of the Killing form we obtain $a^\gamma_{\alpha\beta} = B_{\mathfrak{g}}\big([\xi_\gamma, \eta_\alpha], \eta_\beta\big)$ and consequently

$$\sum_\beta a^\gamma_{\alpha\beta}\xi_\beta = [\xi_\gamma, \eta_\alpha]_-,$$

where $[\cdot,\cdot]_-$ denotes the component of the Lie bracket $[\cdot,\cdot]$ in $\mathfrak{g}_-$. Hence we get for the third term

$$\begin{aligned} \tfrac{1}{2}\textstyle\sum_{\alpha,\beta} B_{\mathfrak{g}}\big(\varphi(\xi_\alpha, \xi_\beta), \psi([\eta_\alpha, \eta_\beta])\big) &= \tfrac{1}{2}\textstyle\sum_{\alpha,\beta,\gamma} B_{\mathfrak{g}}\big(\varphi(\xi_\alpha, \xi_\beta), \psi(a^\gamma_{\alpha\beta}\eta_\gamma)\big) \\ &= \tfrac{1}{2}\textstyle\sum_{\alpha,\beta,\gamma} B_{\mathfrak{g}}\big(\varphi(\xi_\alpha, a^\gamma_{\alpha\beta}\xi_\beta), \psi(\eta_\gamma)\big) \\ &= \tfrac{1}{2}\textstyle\sum_{\alpha,\gamma} B_{\mathfrak{g}}\big(\varphi(\xi_\alpha, \sum_\beta a^\gamma_{\alpha\beta}\xi_\beta), \psi(\eta_\gamma)\big) \\ &= \tfrac{1}{2}\textstyle\sum_\gamma B\left(\sum_\alpha \varphi([\eta_\alpha, \xi_\gamma]_-, \xi_\alpha), \psi(\eta_\gamma)\right). \end{aligned}$$

So it follows

$$\begin{aligned} \langle \partial^*\varphi, \psi\rangle &= -\langle \varphi, \partial\psi\rangle \\ &= \textstyle\sum_\beta B_{\mathfrak{g}}\left(\sum_\alpha \big[\eta_\alpha, \varphi(\xi_\alpha, \xi_\beta)\big] + \frac{1}{2}\sum_\alpha \varphi([\eta_\alpha, \xi_\beta]_-, \xi_\alpha), \psi(\eta_\beta)\right) \end{aligned}$$

and we obtain for $\varphi \in C^2(\mathfrak{g}_-, \mathfrak{g})$ and $X \in \mathfrak{g}_-$

$$\partial^*\varphi(X) = \sum_\alpha [\eta_\alpha, \varphi(\xi_\alpha, X)] + \frac{1}{2}\sum_\alpha \varphi([\eta_\alpha, X]_-, \xi_\alpha).$$

Now we want to prove that the codifferential ∂^* and the differential ∂ are adjoint to each other with respect to a special metric.
For this we use the following lemma from [Tan79] (Lemma 1.5).

Lemma 3.6 *For every $|k|$-graded (semi-)simple Lie algebra $\mathfrak{g}$ there exists an involutive Lie algebra automorphism $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$ which is conjugate linear in the complex case and linear in the real case, and it satisfies $\sigma(\mathfrak{g}_i) = \mathfrak{g}_{-i}$ and $B_{\mathfrak{g}}(X, \sigma(X)) < 0$ for all $X \in \mathfrak{g}$, $X \neq 0$.*

With this automorphism σ we obtain on $\mathfrak{g}$ a positive definite hermitian product in the complex case and a positive definite inner product in the real case $B^*(X, Y) := -B_{\mathfrak{g}}(X, \sigma(Y))$. Note that the Killing form is compatible with the automorphism σ, inheriting this property from the Lie bracket, that is to say $B_{\mathfrak{g}}(X, Y) = B_{\mathfrak{g}}(\sigma(X), \sigma(Y))$ for all $X, Y \in \mathfrak{g}$. We define a conjugate linear map F between $C^n(\mathfrak{g}_-, \mathfrak{g})$ and $C^n(\mathfrak{p}_+, \mathfrak{g})$ in the following way:

$$\begin{aligned} F : C^n(\mathfrak{g}_-, \mathfrak{g}) &\longrightarrow C^n(\mathfrak{p}_+, \mathfrak{g}) \\ F(\varphi)(Z_1, \dots, Z_n) &:= \sigma\left(\varphi\big(\sigma(Z_1), \dots, \sigma(Z_N)\big)\right). \end{aligned}$$

Since σ and the Lie bracket are compatible, the differential ∂ commutes with F, $\partial \circ F = F \circ \partial$.

$$
\begin{aligned}
& F(\partial\varphi)(X_0,\dots,X_n) \\
= \; & \textstyle\sum_i (-1)^i \sigma\left(\left[\sigma(X_i), \varphi\left(\sigma(X_0),\dots,\widehat{\sigma(X_i)},\dots,\sigma(X_n)\right)\right]\right) \\
& + \textstyle\sum_{i<j} (-1)^{i+j} \sigma\circ\varphi\left(\left[\sigma(X_i),\sigma(X_j)\right], \sigma(X_0),\dots,\widehat{\sigma(X_i)},\dots,\widehat{\sigma(X_j)},\dots,\sigma(X_n)\right) \\
= \; & \textstyle\sum_i (-1)^i \big[\underbrace{\sigma^2(X_i)}_{=X_i}, \sigma\circ\varphi\left(\sigma(X_0),\dots,\widehat{\sigma(X_i)},\dots,\sigma(X_n)\right)\big] \\
& + \textstyle\sum_{i<j} (-1)^{i+j} \sigma\circ\varphi\left(\sigma\left[X_i,X_j\right], \sigma(X_0),\dots,\widehat{\sigma(X_i)},\dots,\widehat{\sigma(X_j)},\dots,\sigma(X_n)\right) \\
= \; & \textstyle\sum_i (-1)^i \big[X_i, F(\varphi)(X_0,\dots,\widehat{X_i},\dots,X_n)\big] \\
& + \textstyle\sum_{i<j} (-1)^{i+j} F(\varphi)\big([X_i,X_j], X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_n\big) \\
= \; & \partial F(\varphi)
\end{aligned}
$$

The hermitian (or inner) product B^* on $\mathfrak{g}$ induces a hermitian (or inner) product B^* on $C^n(\mathfrak{g}_-,\mathfrak{g})$ via

$$B^*(\varphi,\psi) := \frac{1}{n!}\sum_{\alpha_1,\dots,\alpha_n} B^*\big(\varphi(\xi_{\alpha_1},\dots,\xi_{\alpha_n}), \psi(\xi_{\alpha_1},\dots,\xi_{\alpha_n})\big),$$

where $\{\xi_\alpha\}$ is a unitary (or orthonormal) basis with respect to B^*.

Proposition 3.2 *The differential $\partial : C^n(\mathfrak{g}_-,\mathfrak{g}) \longrightarrow C^{n+1}(\mathfrak{g}_-,\mathfrak{g})$ and the codifferential $\partial^* : C^{n+1}(\mathfrak{g}_-,\mathfrak{g}) \longrightarrow C^n(\mathfrak{g}_-,\mathfrak{g})$ are adjoint to each other with respect to B^*, i.e.*

$$B^*(\partial\varphi,\psi) = B^*(\varphi,\partial^*\psi)$$

for all $\varphi \in C^n(\mathfrak{g}_-,\mathfrak{g})$ and all $\psi \in C^{n+1}(\mathfrak{g}_-,\mathfrak{g})$.
Especially $C^n_l(\mathfrak{g}_-,\mathfrak{g})$ splits for all n and l into the direct sum of the image of ∂ and the kernel of ∂^*

$$C^n_l(\mathfrak{g}_-,\mathfrak{g}) = \partial\big(C^{n-1}_l(\mathfrak{g}_-,\mathfrak{g})\big) \oplus Ker\left(\partial^*|_{C^n_l(\mathfrak{g}_-,\mathfrak{g})}\right).$$

Further each cohomology class contains a uniquely defined representative which is harmonic (that is ∂-closed and ∂^*-closed).

Proof: To see that ∂ and ∂^* are adjoint to each other we use the dual pairing $\langle\cdot,\cdot\rangle$ of $C^n(\mathfrak{g}_-,\mathfrak{g})$ and $C^n(\mathfrak{p}_+,\mathfrak{g})$ as above. Let $\{\xi_\alpha\}$ be a unitary (or orthonormal) basis of $\mathfrak{g}_-$ with respect to B^*. The corresponding dual basis is again denoted by $\{\eta_\alpha\}$ and we have $\eta_\alpha = -\sigma(\xi_\alpha)$ due to $B_{\mathfrak{g}}(\xi_\alpha, -\sigma(\xi_\alpha)) = B^*(\xi_\alpha,\xi_\alpha) = 1$. Let $\varphi,\psi \in C^n(\mathfrak{g}_-,\mathfrak{g})$ and $F : C^n(\mathfrak{g}_-,\mathfrak{g}) \longrightarrow C^n(\mathfrak{p}_+,\mathfrak{g})$ be the same as above. Then it holds:

$$
\begin{aligned}
(-1)^{n+1}\langle\varphi, F(\psi)\rangle &= (-1)^{n+1}\tfrac{1}{n!}\textstyle\sum_{\alpha_1,\dots,\alpha_n} B\big(\varphi(\xi_{\alpha_1},\dots,\xi_{\alpha_n}), F(\psi)(\eta_{\alpha_1},\dots,\eta_{\alpha_n})\big) \\
&= \tfrac{(-1)^{n+1}}{n!}\textstyle\sum_{\alpha_1,\dots,\alpha_n} B\big(\varphi(\xi_{\alpha_1},\dots,\xi_{\alpha_n}), \sigma\circ\psi(\underbrace{\sigma\eta_{\alpha_1}}_{=-\xi_{\alpha_1}},\dots,\underbrace{\sigma\eta_{\alpha_n}}_{=-\xi_{\alpha_1}})\big) \\
&= \tfrac{1}{n!}\textstyle\sum_{\alpha_1,\dots,\alpha_n} -B\left(\varphi(\xi_{\alpha_1},\dots,\xi_{\alpha_n}), \sigma\circ\psi(\xi_{\alpha_1},\dots,\xi_{\alpha_n})\right) \\
&= \tfrac{1}{n!}\textstyle\sum_{\alpha_1,\dots,\alpha_n} B^*\left(\varphi(\xi_{\alpha_1},\dots,\xi_{\alpha_n}), \psi(\xi_{\alpha_1},\dots,\xi_{\alpha_n})\right) \\
&= B^*(\varphi,\psi).
\end{aligned}
$$

Hence we obtain for $\varphi \in C^n(\mathfrak{g}_-, \mathfrak{g})$ and $\psi \in C^{n-1}(\mathfrak{g}_-, \mathfrak{g})$ using the definition of ∂^* as the negative dual map of the differential $\partial : C^{n-1}(\mathfrak{p}_+, \mathfrak{g}) \longrightarrow C^n(\mathfrak{p}_+, \mathfrak{g})$

$$\begin{aligned} B^*(\varphi, \partial\psi) &= (-1)^{n+1}\langle\varphi, F(\partial\psi)\rangle \\ &= (-1)^{n+1}\langle\varphi, \partial F(\psi)\rangle \\ &= (-1)^n\langle\partial^*\varphi, F(\psi)\rangle \\ &= B^*(\partial^*\varphi, \psi). \end{aligned}$$

Consequently the differential $\partial : C^n(\mathfrak{g}_-, \mathfrak{g}) \longrightarrow C^{n+1}(\mathfrak{g}_-, \mathfrak{g})$ and the codifferential $\partial^* : C^{n+1}(\mathfrak{g}_-, \mathfrak{g}) \longrightarrow C^n(\mathfrak{g}_-, \mathfrak{g})$ are adjoint to each other with respect to B^*, which implies as usual the splitting $C^n_l(\mathfrak{g}_-, \mathfrak{g}) = \partial\big(C^{n-1}_l(\mathfrak{g}_-, \mathfrak{g})\big) \oplus Ker\left(\partial^*|_{C^n_l(\mathfrak{g}_-,\mathfrak{g})}\right)$ and the existence of a uniquely defined harmonic representative in each cohomology class.

□

This will be needed to single out the prolonged bundle in the prolongation procedure later on.

3.4 Cartan Geometry

Now we will explain the basic structures of Cartan geometries which will later on allow the definition of the Cartan boundary for a broad variety of geometric settings. This section is based on [Sha97] and [CS09].

Definition 3.9 *Let $P \subset G$ be a Lie subgroup in a Lie group G. A Cartan geometry of type (G, P) on a manifold M is a principal bundle $\pi : \mathcal{G} \longrightarrow M$ with structure group P together with a $\mathfrak{g} = LA(G)$-valued one-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ satisfying*

- *ω is P-equivariant, i.e. $R_p^*\omega = Ad(p^{-1}) \circ \omega$ for all $p \in P$.*
- *ω reproduces the generators of the fundamental vector fields, that is $\omega(\widetilde{X}) = X$ for all $X \in \mathfrak{p} = LA(P)$.*
- *For every $u \in \mathcal{G}$ the linear map $\omega_u : T_u\mathcal{G} \longrightarrow \mathfrak{g}$ is an isomorphism.*

The one-form ω is called a Cartan connection. Its curvature is defined to be the $\mathfrak{g}$-valued two-form $\Omega := d\omega + \frac{1}{2}[\omega, \omega]^\wedge \in \Omega^2(\mathcal{G}, \mathfrak{g})$. Here $[\cdot, \cdot]^\wedge$ denotes the brackets for forms in $\Omega^*(M, \mathfrak{g})$.

$$\begin{array}{rcll} [\cdot,\cdot]^\wedge : \Omega^p(M,\mathfrak{g}) \times \Omega^q(M,\mathfrak{g}) & \longrightarrow & \Omega^{p+q}(M,\mathfrak{g}) & \\ (\omega, \tau) & \mapsto & [\omega,\tau]^\wedge & = \sum_{i,j}(\omega_i \wedge \tau_j)[a_i, a_j]_{\mathfrak{g}} \\ & & & \text{where } (a_i)_{i=1}^r \text{ is a basis of } \mathfrak{g} \\ & & & \text{and } \omega = \sum_i \omega_i a_i,\ \tau = \sum_j \tau_j a_j \\ & & & \text{with } \omega_i \in \Omega^p(M,\mathbb{R}),\ \tau_j \in \Omega^q(M,\mathbb{R}) \end{array}$$

A Cartan connection ω is called flat, if its curvature vanishes.

Definition 3.10 *Let $(\mathcal{G}_1, \pi_1, M_1; \omega_1)$ and $(\mathcal{G}_2, \pi_2, M_2; \omega_2)$ be two Cartan geometries of type (G, P). Let $f : M_1 \longrightarrow M_2$ be an immersion covered by a P-bundle map $\widetilde{f} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ satisfying $\widetilde{f}^*\omega_2 = \omega_1$. Then f is called a local isomorphism of the Cartan geometries. If in addition f is a diffeomorphism it is called an isomorphism of the Cartan geometries.*

Proposition 3.3 (Theorem 5.1 in [Sha97]) *Let $(\mathcal{G}, \pi, M; \omega)$ be a Cartan geometry of type (G, P). If the Cartan connection ω is flat, that is to say $\Omega^\omega = 0$, the Cartan geometry is locally isomorphic to the homogeneous model $(G, \pi, P; \omega_G)$.*

Lemma 3.7 *Let $(\mathcal{G},\pi,M,\omega)$ be a Cartan geometry of type (G,P). Then there is a canonical bundle isomorphism*

$$TM \simeq \mathcal{G}\times_P \mathfrak{g}/_{\mathfrak{p}}.$$

Proof: For any $u\in\mathcal{G}$ and $x:=\pi(u)$ we have a canonical isomorphism $\varphi_u : T_xM \longrightarrow \mathfrak{g}/_{\mathfrak{p}}$ which makes the following diagram commute.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & ker(d\pi_u) & \hookrightarrow & T_u\mathcal{G} & \overset{d\pi_u}{\twoheadrightarrow} & T_xM & \longrightarrow & 0\\
 & & \sim\downarrow \omega_u & \circlearrowleft & \sim\downarrow\omega_u & \circlearrowleft & \sim\downarrow\varphi_u & & \\
0 & \longrightarrow & \mathfrak{p} & \hookrightarrow & \mathfrak{g} & \overset{pr}{\twoheadrightarrow} & \mathfrak{g}/_{\mathfrak{p}} & \longrightarrow & 0
\end{array}$$

The isomorphism φ_u is given by

$$\varphi_u(X) := pr\circ\omega_u(Y) \text{ with } Y\in T_u\mathcal{G} \text{ such that } d\pi(Y)=X\in T_xM.$$

This is well defined since for two vectors $Y_1,Y_2\in T_u\mathcal{G}$ projecting onto the same vector $d\pi(Y_1)=d\pi(Y_2)=X$ the difference Y_1-Y_2 is a vertical vector and therefore $\omega_u(Y_1-Y_2)\in\mathfrak{p}$ and $pr\circ\omega_u(Y_1-Y_2)=0$.
Given $p\in P$ we can write for φ_{R_pu}

$$\begin{aligned}
\varphi_{R_pu}(X) &= pr\circ\omega_{R_pu}(dR_pY)\\
&= pr\circ Ad(p^{-1})\circ\omega_u(Y)\\
&= Ad(p^{-1})\circ\varphi_u(X).
\end{aligned}$$

Hence we have $\varphi_{R_pu}=Ad(p^{-1})\circ\varphi_u$. So we have a smooth map between the tangent bundle over M and $\mathcal{G}\times\mathfrak{g}$.

$$\begin{aligned}
\phi:\mathcal{G}\times\mathfrak{g} &\longrightarrow TM\\
(u,X) &\mapsto \varphi_u^{-1}\circ pr(X)\in T_{\pi(u)}M
\end{aligned}$$

For $p\in P$ it holds

$$\begin{aligned}
\phi(R_pu, Ad(p^{-1})X) &= \underbrace{\varphi_{R_pu}^{-1}}_{\varphi_u^{-1}\circ Ad(p)}\circ pr\circ Ad(p^{-1})(X)\\
&= \varphi^{-1}\circ pr(X)\\
&= \phi(u,X).
\end{aligned}$$

Consequently the map ϕ induces canonically the smooth bundle map $\phi:\mathcal{G}\times_P\mathfrak{g}/_{\mathfrak{p}}\longrightarrow TM$. This is an isomorphism on the fibres and covers the identity. Hence this is a vector bundle isomorphism.

□

The curvature of a Cartan connection has similar properties as the curvature of principal bundle connections.

Lemma 3.8 *Let $(\mathcal{G},\pi,M;\omega)$ be a Cartan geometry of type (G,P). If $\mu:\mathcal{G}\longrightarrow P$ is a smooth map and $F:\mathcal{G}\longrightarrow\mathcal{G}$ defined by $F(u)=R_{\mu(u)}u$, then the curvature of the Cartan connection satisfies*

$$F^*\Omega^\omega = \Omega^{F^*\omega} = Ad(\mu(\cdot)^{-1})\cdot\Omega^\omega.$$

The curvature of a Cartan connection is P-equivariant and horizontal. Furthermore the Bianchi identity holds, $d\Omega^\omega = [\Omega^\omega,\omega]^\wedge$.

For $X \in \mathfrak{g}$ the term $\omega^{-1}(X)$ denotes the ω-constant vector field generated by X,

$$\omega^{-1}(X) : \mathcal{G} \ni u \mapsto \omega^{-1}(X)(u) := \left(\omega|_{T_u\mathcal{G}}\right)^{-1}(X) \in T_u\mathcal{G}.$$

According to the definition of the curvature we can write for $X, Y \in \mathfrak{g}$

$$\begin{aligned}
\Omega^\omega(\omega^{-1}(X), \omega^{-1}(Y)) &= \tfrac{1}{2}[\omega,\omega]^\wedge(\omega^{-1}(X), \omega^{-1}(Y)) + d\omega(\omega^{-1}(X), \omega^{-1}(Y)) \\
&= [X,Y]_{\mathfrak{g}} + \omega^{-1}(X)\big(\underbrace{\omega(\omega^{-1}(Y))}_{=Y=const}\big) - \omega^{-1}(Y)\big(\underbrace{\omega(\omega^{-1}(X))}_{=X=const}\big) \\
&\quad -\omega([\omega^{-1}(X), \omega^{-1}(Y)]) \\
&= [X,Y]_{\mathfrak{g}} - \omega([\omega^{-1}(X), \omega^{-1}(Y)]).
\end{aligned}$$

I.e. the curvature describes the difference between both commutators.
If the Lie algebra $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ is endowed with a grading the curvature of a Cartan connection splits according to this grading, $\Omega_i^\omega := pr_{\mathfrak{g}_i} \circ \Omega^\omega$. The $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$-component of the curvature is called the torsion of the Cartan connection ω,

$$\tau^\omega := pr_{\mathfrak{g}_-} \circ \Omega^\omega.$$

Another way of splitting the curvature uses the homogeneity. $\Omega^\omega = \sum_{l=-k+2}^{3k} (\Omega^\omega)^{(l)}$ with $(\Omega^\omega)^{(l)} : \mathfrak{g}_i \times \mathfrak{g}_j \longrightarrow \mathfrak{g}_{i+j+l}$.

Definition 3.11 *A Cartan geometry $(\mathcal{G}, \pi, M; \omega)$ with curvature Ω^ω is called*
flat, if its curvature vanishes, $\Omega^\omega = 0$,
torsion free, if its torsion is zero, $\tau^\omega = 0$,
normal, if the codifferential of its curvature vanishes, $\partial^* \circ \Omega^\omega = 0$, or
regular, if it is normal and its components $(\Omega^\omega)^{(l)}$ vanish for all $l \leq 0$.

3.5 Parabolic Geometry

Having given some manifold endowed with a certain geometric structure we somehow have to construct a suitable Cartan bundle and a Cartan connection. One very general way to achieve this uses the special properties of parabolic geometries. Since both conformal and CR geometries can be dealt with in this way, we will now discuss the construction of a Cartan bundle and a Cartan connection for parabolic geometries from [CS00]. Although it is quite complicated and far more general, than needed here, we will stick to the general setting first, because we consider this general construction form [CS00] very interesting with a high potential of usage. Later on we will restrict to CR and conformal geometry.
A parabolic geometry is a Cartan geometry $(\mathcal{G}, \pi, M; \omega)$ of type (G, P) such that G is semisimple and P parabolic, i.e. $\mathfrak{p} = LA(P)$ contains a Borel subalgebra (a maximal solvable subalgebra). For certain manifolds a general construction resulting in a Cartan bundle endowed with a Cartan connection will be given in this section. This construction is motivated by subbundles and structures that can be derived from the Cartan geometry. So we will at first explain those structures and then build up the Cartan bundle step by step. We will do this by starting with a semisimple $|k|$-graded Lie algebra and a manifold with a corresponding filtration of the tangent bundle. Then a P-frame bundle of degree l consisting of a $P/_{P_+^l}$-principal bundle and a frame form of length l which satisfies the structure equations will be introduced. Next the torsion of a frame form is defined. This will enable us to single out the P-frame bundles which are harmonic.
For the prolongation of a harmonic P-frame bundle we will construct an enlarged bundle endowed with a $P/_{P_+^{l+1}}$-action and define a natural analog of a frame form. Then we will

extract the elements with ∂^*-closed torsion to define a harmonic P-frame bundle of degree $l+1$ presuming the vanishing of the first cohomology group of degree l. Thus iterated application of this construction will build up a Cartan bundle endowed with a Cartan connection. Finally we will address the question of uniqueness.

3.5.1 Motivation

Let us at first take a closer look at the structures of a parabolic geometry. This will outline the essential ingredients for the construction of a parabolic geometry for certain manifolds, such as CR, conformal and many more, later on.
Assume we have given a parabolic geometry $(\mathcal{G}, \pi, M; \omega)$ of type (G, P). The Lie algebra of the Lie group G is $|k|$-graded, $LA(G) = \mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ and with the subalgebras $\mathfrak{p} := \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ and $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ we can define several subgroups

- $P_+ = \exp(\mathfrak{p}_+)$,
- $P_+^l = \exp(\mathfrak{p}_+^l) = \exp(\mathfrak{g}_l \oplus \cdots \oplus \mathfrak{g}_k)$ and
- $G_l := P/_{P_+^{l+1}} = G_0 \ltimes P_+/_{P_+^{l+1}}$ for $l = 0, \ldots, k$ and we have $G_k = P$.

With the help of these subgroups we define the subbundles $\mathcal{G}_l := \mathcal{G}/_{P_+^{l+1}}$ for $l = 0, \ldots, k$ and obtain the principal G_l-bundle $\pi_l : \mathcal{G}_l \longrightarrow M$.

P-bundle

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "\pi_+^{l+1}"] \arrow[rr, bend left, "\pi"] & \mathcal{G}_l \arrow[r, "\pi_l"] & M
\end{tikzcd}$$

P_+^{l+1}-bundle $\qquad$ G_l-bundle

A filtration of $T\mathcal{G}$ is obtained with the help of the Cartan connection ω.

$$T^i\mathcal{G} := \omega^{-1}(\mathfrak{g}^i) = \omega^{-1}(\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k)$$

$$T\mathcal{G} = T^{-k}\mathcal{G} \supset T^{-k+1}\mathcal{G} \supset \cdots \supset T^0\mathcal{G} = Tv\mathcal{G} \supset \cdots \supset T^k\mathcal{G} \supset \{0\}$$

Note that for $i \geq 0$ the subbundle $T^i\mathcal{G} = \widetilde{\mathfrak{g}^i}$ is the subbundle of the fundamental vector fields $\widetilde{X}$ with $X \in \mathfrak{g}^i = \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$.
Using the projection we get a filtration for $T\mathcal{G}_l$, $l \in \{0, \ldots, k\}$ by setting for $i \in \{-k, \ldots, l\}$

$$T^i\mathcal{G}_l := (d\pi_+^{l+1})(T^i\mathcal{G}).$$

For $i \in \{-k, \ldots, -1\}$ we define $\theta_i^k \in \Gamma\big((T^i\mathcal{G})^* \otimes \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+k}\big)$ using the Cartan connection and projection:

$$\begin{tikzcd}
T_u^i\mathcal{G} \arrow[r, "\omega_u|_{T_u^i\mathcal{G}}"] \arrow[dr, "\theta_i^k(u)"'] & \mathfrak{g}^i = \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k \arrow[d, "\mathrm{proj}"] \arrow[dl, phantom, "\circlearrowright"] \\
 & \mathfrak{g}^i/_{\mathfrak{g}^{i+k+1}} \simeq \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+k}
\end{tikzcd}$$

$$\theta_i^k = proj \circ \omega.$$

It holds $ker\left(\theta_i^k\right) = T^{i+k+1}\mathcal{G}$. Furthermore θ_i^k inherits the P-equivariance from the Cartan connection ω and reproduces the generators of the fundamental vector fields. We want to generalize this definition of θ_i^k for other indices.

Lemma 3.9 *For a parabolic geometry $(\mathcal{G}, \pi, M; \omega)$ of type (G, P) with a $|k|$-grading of the Lie algebra $\mathfrak{g}$ there is for all indices $i \in \{-k, \dots, -1\}$ and $l \in \{0, \dots, k\}$ a uniquely defined section $\theta_i^l \in \Gamma\big((T^i\mathcal{G}_l)^* \otimes \mathfrak{g}_i \oplus \dots \oplus \mathfrak{g}_{i+l}\big)$ with $ker\big(\theta_i^l(\pi_+^{l+1}(u))\big) = T^{i+l+1}_{\pi_+^{l+1}(u)}\mathcal{G}_l$ to make the following diagram commute.*

$$\begin{tikzcd}
T_u^i\mathcal{G} \arrow[rr, "\theta_i^k(u)"] \arrow[d, "d\pi_+^{l+1}"'] \arrow[drr, phantom, "\circlearrowright"] & & \mathfrak{g}_i \oplus \dots \oplus \mathfrak{g}_{i+k} \arrow[d] \\
T^i_{\pi_+^{l+1}(u)}\mathcal{G}_l \arrow[rr, "\theta_i^l(\pi_+^{l+1}(u))"'] & & \mathfrak{g}_i \oplus \dots \oplus \mathfrak{g}_{i+l}
\end{tikzcd}$$

Note that for $l = k$ this is θ_i^k as defined above.

Proof:

Keeping in mind that we have $l \in \{0, \dots, k\}$ and $i \in \{-k, \dots, -1\}$ it certainly holds $T_u^i\mathcal{G}/_{T_u^{i+l+1}\mathcal{G}} \simeq T^i_{\pi_+^{l+1}(u)}\mathcal{G}_l/_{T^{i+l+1}_{\pi_+^{l+1}(u)}\mathcal{G}_l}$. To prove the existence of $\theta_i^l \in \Gamma\big((T^i\mathcal{G}_l)^* \otimes \mathfrak{g}_i \oplus \dots \oplus \mathfrak{g}_{i+l}\big)$ we consider the following diagram of exact sequences. Recall that the kernel of $\theta_i^k(u)$ is $T_u^{i+k+1}\mathcal{G}$. To simplify the notation we set $\widetilde{u} := \pi_+^{l+1}(u)$.

$$\begin{tikzcd}
& & & T_u^i\mathcal{G}/_{T_u^{i+l+1}\mathcal{G}} \arrow[d, leftrightarrow, "\sim"] & \\
0 \arrow[r] & T_u^{i+l+1}\mathcal{G}/_{T_u^{i+k+1}\mathcal{G}} \arrow[r, hook] \arrow[d, "proj \circ \omega_u"] & T_u^i\mathcal{G}/_{T_u^{i+k+1}\mathcal{G}} \arrow[r, two heads] \arrow[d, "\theta_i^k(u)"] & T_{\widetilde{u}}^i\mathcal{G}_l/_{T_{\widetilde{u}}^{i+k+1}\mathcal{G}_l} \arrow[r] \arrow[d, dashed, "\exists!\widetilde{\theta_i^l}(\widetilde{u})"] & 0 \\
0 \arrow[r] & \mathfrak{g}^{i+l+1}/_{\mathfrak{g}^{i+k+1}} \arrow[r, hook] & \mathfrak{g}^i/_{\mathfrak{g}^{i+k+1}} \arrow[r, two heads] & \mathfrak{g}^i/_{\mathfrak{g}^{i+k+1}} \arrow[r] & 0
\end{tikzcd}$$

The uniquely given map $\widetilde{\theta_i^l}(\widetilde{u})$ induces the map $\theta_i^l(\widetilde{u}) : T_{\widetilde{u}}^i\mathcal{G}_l \longrightarrow \mathfrak{g}^i/_{\mathfrak{g}^{i+l+1}}$ via

$$\theta_i^l(\widetilde{u})\big(X\big) := \widetilde{\theta_i^l}(\widetilde{u})\big([X]\big) \text{ for } X \in T_{\widetilde{u}}^i\mathcal{G}_l \text{ and } [X] = X + T_{\widetilde{u}}^{i+l+1}\mathcal{G}_l \in T_{\widetilde{u}}^i\mathcal{G}_l/_{T_{\widetilde{u}}^{i+l+1}\mathcal{G}_l}.$$

So according to the definition we obtain $ker\big(\theta_i^l(\widetilde{u})\big) = T_{\widetilde{u}}^{i+l+1}\mathcal{G}_l$ as wanted. Furthermore the map θ_i^l inherits the P-equivariance from the Cartan connection ω, i.e. for all elements $p \in P$ we have $(R_p^*\theta_i^l)_u = \theta_i^l(R_p\widetilde{u}) \circ dR_p = Ad(p^{-1}) \circ \theta_i^l(\widetilde{u})$. And since the action of P_+^{l+1} is trivial on $\mathfrak{g}^i/_{\mathfrak{g}^{i+l+1}}$ we actually have P_+^{l+1}-invariance for θ_i^l, that is $R_p^*\theta_i^l = \theta_i^l$ for all $p \in P_+^{l+1}$. Hence θ_i^l is well-defined. And from the diagram above the wanted property

$$\theta_i^l\big(\pi_+^{l+1}(u)\big) \circ d\pi_+^{l+1} = proj_{\mathfrak{g}_i \oplus \dots \oplus \mathfrak{g}_{i+l}} \circ \theta_i^k(u)$$

follows directly.

□

θ_i^l is G_l-equivariant and reproduces the generators of the fundamental vector fields. Later on the definition of a frame form will be given, inspired by $\theta^l := (\theta_{-k}^l, \dots, \theta_{-1}^l)$, which will be a frame form of length $l+1$ on $\mathcal{G}_l$.

Remark 3.1 *All holds for $l = 0$ if we set $P^0_+ := P$ and $\pi^0_+ := \pi$. Then we have $\mathcal{G}_{-1} = M$ and we obtain a filtration of TM*

$$TM = T^{-k}M \supset \cdots \supset T^{-1}M \supset \{0\}$$

with $rank(T^iM) = dim(\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{-1})$ for $i \in \{-k, \ldots, -1\}$. Moreover the associated graded tangent bundle is given by

$$Gr_xTM := \underbrace{\left(T_x^{-k}M/_{T_x^{-k+1}M}\right)}_{=:Gr_{-k}(T_xM)} \oplus \underbrace{\left(T_x^{-k+1}M/_{T_x^{-k+2}M}\right)}_{=:Gr_{-k+1}(T_xM)} \oplus \cdots \oplus \underbrace{T^{-1}M}_{=:Gr_{-1}(T_xM)} .$$

Its frame bundle is

$$\mathcal{F}_x\big(Gr(TM)\big) := \left\{ (\varphi_{-k}, \ldots, \varphi_{-1})_x \;\middle|\; \begin{array}{l} \varphi_i : T_x^iM \longrightarrow \mathfrak{g}_i, \\ \text{with } ker(\varphi_i) = T_x^{i+1}M \end{array} \right\}.$$

Since $\theta^0(u) = \big(\theta^0_{-k}(u), \ldots, \theta^0_{-1}(u)\big)$ is an associated graded repere for $u \in \mathcal{G}_0$ the frame bundle of the associated graded tangent bundle can be reduced to the structure group G_0.

The structures we have identified for a parabolic geometry will now be used to construct a Cartan bundle. Starting with a given harmonic G_0-principal bundle endowed with a frame form of length one, i.e. a harmonic P-frame bundle of degree one, we will prolong this bundle to a harmonic P-frame bundle of degree two. Iterated application of the prolongation procedure will finally result in a harmonic P-frame bundle of degree $2k+1$, which is a Cartan bundle endowed with a Cartan connection.

Thus first of all we have to explain P-frame bundles and identify the harmonic ones by introducing the torsion of such bundles. Then the actual prolongation can start by defining the enlarged bundle $\hat{E}$. On this bundle we have a natural analog of a frame form. However the bundle $\hat{E}$ is “to large”. Again the torsion will be the mean to identify the subbundle of $\hat{E}$ needed, yielding the prolonged bundle - a P-frame bundle of degree $l+1$.

3.5.2 P-Frame Bundles

Let G be a Lie group with semisimple $|k|$-graded Lie algebra $\mathfrak{g}$. Furthermore let M be a manifold with a filtration of the tangent bundle $TM = T^{-k}M \supset T^{-k+1}M \supset \cdots \supset T^{-1}M$ by vector subbundles such that the rank of T^jM equals the dimension of $\mathfrak{g}^j/_{\mathfrak{g}^0} = \mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_{-1}$ for every $j = -k, \ldots, -1$.

Given a $P/_{P^i_+}$-principal bundle $p : E \longrightarrow M$ for some $i = 1, \ldots, k$ we obtain an induced filtration via

$$\begin{array}{rcll} T^jE & := & dp^{-1}(T^jM), & j = -k, \ldots, -1, \\ T^0E & := & TvE, & \text{the vertical bundle of } E \text{ and} \\ T^jE & := & \{\widetilde{X} \mid X \in \mathfrak{g}^j/_{\mathfrak{g}^i} = \mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_{i-1}\}, & j = 1, \ldots, i-1. \end{array}$$

The induced action dR of $P/_{P^i_+}$ on the tangent bundle TE respects its filtration.

Definition 3.12 *If E is a $P/_{P_+^i}$-principal bundle let $l \in \{1,\dots,i\}$, for E being a P-principal bundle let $l \in \{1,\dots,2k+1\}$. A frame form of length l on E is a k-tuple $\theta = (\theta_{-k},\dots,\theta_{-1})$ with $\theta_j : T^jE \longrightarrow \mathfrak{g}_j \oplus \dots \oplus \mathfrak{g}_{j+l-1} = \mathfrak{g}^j/_{\mathfrak{g}^{j+l}}$ being a smooth section of the bundle of linear maps $L(T^jE, \mathfrak{g}^j/_{\mathfrak{g}^{j+l}})$ such that*

1. $ker\theta_j|_{T_u^jE} = T_u^{j+l}E$,
2. $pr_{\mathfrak{g}_j} \circ \theta_j|_{T^{j+1}E} \equiv 0$ *and* $pr_{\mathfrak{g}_{j+1}\oplus\dots\oplus\mathfrak{g}_{j+l-1}} \circ \theta_j|_{T^{j+1}E} = pr_{\mathfrak{g}_{j+1}\oplus\dots\oplus\mathfrak{g}_{j+l-1}} \circ \theta_{j+1}$,
3. θ_j *is $P/_{P_+^i}$-equivariant, that is to say* $R_p^*\theta_j = pr_{\mathfrak{g}_j\oplus\dots\oplus\mathfrak{g}_{j+l-1}} \circ Ad(p^{-1}) \circ \theta_j$, *and*
4. *for $X \in \mathfrak{g}^0/_{\mathfrak{g}^i}$ we have*

$$\begin{aligned}\theta_j(\widetilde{X}) &= 0 && \text{for } j+l \le 0,\\ \theta_j(\widetilde{X}) &= pr_{\mathfrak{g}_0\oplus\dots\oplus\mathfrak{g}_{j+l-1}}X && \text{for } j+l > 0.\end{aligned}$$

Remark 3.2

- *Having given a frame form θ of length l we obtain a frame form of length $l-1$ by simply omitting the $\mathfrak{g}_{j+l-1}$ component of each θ_j.*
- *A frame form $\theta = (\theta_{-k},\dots,\theta_{-1})$ of length one is composed of smooth equivariant sections θ_j of $L(T^jE,\mathfrak{g}_j)$ which induce for each element $u \in E$ an isomorphism $T_u^jE/_{T_u^{j+1}E} \xrightarrow{\sim} \mathfrak{g}_j$.*
- *For l greater than $k+2$ some components of the frame form θ are just restrictions of lower components and contain no new information.*
- *For $l = 2k+1$ the whole information is contained in the form θ_{-k} which is a Cartan connection.*

Now let θ be a frame form of length one. I.e. we have a k-tuple $\theta = (\theta_{-k},\dots,\theta_{-1})$ such that the maps $\theta_j : T^jE \longrightarrow \mathfrak{g}_j$ induce isomorphisms $T_u^jE/_{T_u^{j+1}E} \xrightarrow{\sim} \mathfrak{g}_j$. Thus for indicies $i,j \in \{-k,\dots,-1\}$ with $i+j=-k$ and smooth sections $\xi \in \Gamma(T^iE)$ and $\eta \in \Gamma(T^jE)$ we have

$$d\theta_{-k}(\xi,\eta) + [\theta_i(\xi),\theta_j(\eta)]_{\mathfrak{g}} \quad \text{with } i+j=-k$$

is an element of $\mathfrak{g}_{-k}$. The collection of all of those functions is called the structure function of degree $-k$.
Assume that the structure function of degree $-k$ vanishes and choose for some $\widetilde{j} > j = -k-i$ a section $\widetilde{\eta} \in \Gamma(T^{\widetilde{j}}E) \subset \Gamma(T^jE)$. Hence we obtain $\theta_j(\widetilde{\eta}) = 0$ and $\theta_{-k}(\widetilde{\eta}) = \theta_{-k}(\xi) = 0$. So the structure function gives

$$\begin{aligned}0 &= d\theta_{-k}(\xi,\widetilde{\eta}) + [\theta_i(\xi),\underbrace{\theta_j(\widetilde{\eta})}_{=0}]_{\mathfrak{g}}\\ &= \xi\big(\underbrace{\theta_{-k}(\widetilde{\eta})}_{=0}\big) - \widetilde{\eta}\big(\underbrace{\theta_{-k}(\xi)}_{=0}\big) - \theta_{-k}\big([\xi,\widetilde{\eta}]\big)\\ &= -\theta_{-k}\big([\xi,\widetilde{\eta}]\big).\end{aligned}$$

Consequently the commutator $[\xi,\widetilde{\eta}]$ is actually a section of $T^{-k+1}E$. Stated in general the commutator of sections $\xi \in \Gamma(T^iE)$ and $\eta \in \Gamma(T^jE)$ with $i+j > -k$ is a section of $T^{-k+1}E$, $[\xi,\eta] \in \Gamma(T^{-k+1}E)$. If we extend the map $\theta_{-k+1} : T^{-k+1}E \longrightarrow \mathfrak{g}_{-k+1}$ to a one form $\theta_{-k+1} : TE \longrightarrow \mathfrak{g}_{-k+1}$ and take sections $\xi \in \Gamma(T^iE)$, $\eta \in \Gamma(T^jE)$ with $i + j = -k + 1$ we can write

$$\begin{aligned} & d\theta_{-k+1}(\xi, \eta) + \big[\theta_i(\xi), \theta_j(\eta)\big]_{\mathfrak{g}} \\ = \; & \xi\big(\underbrace{\theta_{-k+1}(\eta)}_{=0}\big) - \eta\big(\underbrace{\theta_{-k+1}(\xi)}_{=0}\big) - \theta_{-k+1}\big(\underbrace{[\xi, \eta]}_{\in\Gamma(T^{-k+1}E)}\big) + \big[\theta_i(\xi), \theta_j(\eta)\big]_{\mathfrak{g}} \end{aligned}$$

which is independent of the extension of θ_{-k+1} and therefore well defined. The collection of these functions is called structure function of degree $-k + 1$. Iteration gives the structure functions of degree $-k$ up to -2

$$d\theta_{i+j}(\xi, \eta) + [\theta_i(\xi), \theta_j(\eta)]_{\mathfrak{g}} \in \mathfrak{g}_{i+j}.$$

I.e. the vanishing of the structure function of degree μ implies for ξ and η sections as above with $i + j > \mu$ that their commutator is actually a section of $T^{\mu+1}E$ and thus the structure equation of degree $\mu + 1$ is well defined.

Definition 3.13 *A frame form of length one is said to satisfy the structure equations if and only if the structure functions of all degrees $-k, \ldots, -2$ vanish. A frame form of length l is said to satisfy the structure equations if and only if the underlying frame form of length one has this property.*

As we have just seen, if a frame form satisfies the structure equations we have for the commutator of sections of the subbundle T^iE and T^jE:

$$\big[\Gamma(T^iE), \Gamma(T^jE)\big] \subset \Gamma(T^{i+j}E) \text{ for } i, j < 0.$$

We even obtain

$$\big[\Gamma(T^iE), \Gamma(T^0E)\big] \subset \Gamma(T^iE) \text{ for } i < 0,$$

since we have for $\xi \in \Gamma(T^iE)$ and $A \in \mathfrak{g}^0$ according to the vanishing of the structure function for the frame form of length one

$$\begin{aligned} 0 \; &= \; d\theta_{i-1}(\xi, \widetilde{A}) + [\theta_i(\xi), \underbrace{\theta_{-1}(\widetilde{A})}_{=0}]_{\mathfrak{g}} \\ &= \; \xi\big(\underbrace{\theta_{i-1}(\widetilde{A})}_{=0}\big) - \widetilde{A}\big(\underbrace{\theta_{i-1}(\xi)}_{=0}\big) - \theta_{i-1}([\xi, \widetilde{A}]). \end{aligned}$$

Thus $[\xi, \widetilde{A}]$ is an element of the kernel of θ_{i-1} which is $\Gamma(T^iE)$.

Definition 3.14 *Let $\mathfrak{g}$ be a $|k|$-graded Lie algebra and M a manifold with a filtration of the tangent bundle $TM = T^{-k}M \supset \cdots \supset T^{-1}M$, with $rank(T^iM) = dim(\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{-1})$. For $l = 1, \ldots, 2k + 1$ a P-frame bundle of degree l over M is defined to be a principal bundle $p : E \longrightarrow M$ with structure group $P/_{P_+^l}$ joined by a frame form θ of length l on E, which satisfies the structure equations.*

Thus a P-frame bundle of degree one is a $P/_{P_+} = G_0$-principal bundle and the frame form induces isomorphisms $\theta_j|_{T_u^jE/_{T_u^{j+1}E}} : T_u^jE/_{T_u^{j+1}E} \xrightarrow{\sim} \mathfrak{g}_j$. Since the vertical bundle TvE is contained in the subbundle $T^{j+1}E$ the subspace $\mathfrak{g}_j$ is also isomorphic to $T_x^jM/_{T_x^{j+1}M}$. So a P-frame bundle of degree one is actually a G_0-reduction of the associated graded vector bundle $T^{-k}M/_{T^{-k+1}M} \oplus \cdots \oplus T^{-2}M/_{T^{-1}M} \oplus T^{-1}M$ to the tangent bundle of M, which satisfies the structure equations. A P-frame bundle of degree $2k + 1$ is just a P-principal bundle endowed with a Cartan connection which satisfies the structure equations.

3.5.3 Harmonic P-Frame Bundles

We now distinguish the harmonic P-frame bundles which will be the ones to be prolonged.

Lemma 3.10 *Let θ be a frame form of length l. For elements $\xi \in T^jE$ with $j < 0$ and $A \in \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{l-1}$ it holds*

$$d\theta_j(\widetilde{A}, \xi) = -pr_{\mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_{j+l-1}}[A, \theta_j(\xi)]_{\mathfrak{g}}.$$

Proof: Let $\widetilde{\theta}_j : TE \longrightarrow \mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_{j+l-1}$ be a smooth global extension of the frame form $\theta_j : T^jE \longrightarrow \mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_{j+l-1}$. The term $d\widetilde{\theta}_j(\widetilde{A}, \xi)$ is actually independent of the extension of θ_j since

$$\begin{aligned} d\widetilde{\theta}_j(\widetilde{A}, \xi) &= \widetilde{A}\big(\underbrace{\widetilde{\theta}_j(\xi)}_{=\theta_j(\xi)}\big) - \xi\big(\widetilde{\theta}_j(\underbrace{\quad\widetilde{A}\quad}_{\in T^0E \subset T^jE})\big) - \widetilde{\theta}_j\big(\underbrace{[\widetilde{A}, \xi]}_{\in T^jE}\big) \\ &= \widetilde{A}\big(\theta_j(\xi)\big) - \xi\big(\theta_j(\widetilde{A})\big) - \theta_j\big([\widetilde{A}, \xi]\big) \\ &= d\theta_j(\widetilde{A}, \xi). \end{aligned}$$

Keeping in mind that $\widetilde{\theta}_j(\widetilde{A}) = \theta_j(\widetilde{A}) = const$, we can write with the help of the Lie derivative and using the equivariancy of the frame form

$$\begin{aligned} d\theta_j(\widetilde{A}, \xi) &= d\widetilde{\theta}_j(\widetilde{A}, \xi) \\ &= \big(d(\underbrace{i_{\widetilde{A}}\widetilde{\theta}_j}_{\equiv const}) + i_{\widetilde{A}}(d\widetilde{\theta}_j)\big)(\xi) \\ &= L_{\widetilde{A}}\widetilde{\theta}_j(\xi) \\ &= \tfrac{d}{dt}\Big((R^*_{\exp(tA)}\widetilde{\theta}_j)(\xi)\Big)\Big|_{t=0} \\ &= \tfrac{d}{dt}\Big((R^*_{\exp(tA)}\theta_j)(\xi)\Big)\Big|_{t=0} \\ &= \tfrac{d}{dt}\big(pr_{\mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_{j+l-1}} \circ Ad(\exp(-tA)) \circ \theta_j(\xi)\big)\big|_{t=0} \\ &= -pr_{\mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_{j+l-1}} \circ ad(A) \circ \theta_j(\xi) \\ &= -pr_{\mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_{j+l-1}}[A, \theta_j(\xi)]_{\mathfrak{g}}. \end{aligned}$$

However this is the statement claimed.

□

We can now define the torsion of a P-frame bundle $(E, p, M; \theta)$ of degree l. Let u be a point in E. For $i, j < 0$ and $X \in \mathfrak{g}_i$, $Y \in \mathfrak{g}_j$ choose vectors $\xi \in T^i_uE$ and $\eta \in T^j_uE$ with $\theta_i(\xi) = X$ and $\theta_j(\eta) = Y$. We set

$$t_\theta(u)(X, Y) := \begin{cases} pr_{\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{i+j+l-1}} \circ d\theta_{-k}(\xi, \eta), & \text{if } i + j < -k \\ d\theta_{i+j}(\xi, \eta), & \text{if } i + j \geq -k \end{cases}.$$

This is an element of $\mathfrak{g}_{i+j} \oplus \cdots \oplus \mathfrak{g}_{i+j+l-1}$. Thus we have a linear map $t_\theta(u) : \mathfrak{g}_- \wedge \mathfrak{g}_- \longrightarrow \mathfrak{g}$ which has homogeneous components solely of degrees $0, \ldots, l-1$. However we have to prove that this is well defined, i.e. independent of the choice of ξ and η.
Having given $\xi_1, \xi_2 \in T^i_uE$ with $\theta_i(\xi_1) = \theta_i(\xi_2) = X$ the difference $\xi_1 - \xi_2$ is an element of the kernel of θ_i which is T^{i+l}_uE.

first case $i + l < 0$

In this case we have $\eta \in T^j_uE \subset E^{i+j+l}_uE$, $\xi_1 - \xi_2 \in T^{i+l}_uE \subset T^{i+j+l}_uE$ and consequently $[\eta, \xi_1 - \xi_2] \in T^{i+j+l}_uE$. Thus for $i + j \geq -k$ we obtain $d\theta_{i+j}(\xi_1 - \xi_2, \eta) \equiv 0$ and for $i + j < -k$ the term $pr_{\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{i+j+l-1}} \circ d\theta_{-k}(\xi_1 - \xi_2, \eta)$ vanishes since the critical components of $\theta_{-k}(\eta)$ and $\theta_{-k}([\xi_1-\xi_2,\eta])$ correspond to each of the first components of $\theta_{-k},\dots,\theta_{i+j+l-1}$ which are zero. I.e.

$$\begin{aligned}
& pr_{\mathfrak{g}_{-k}\oplus\cdots\oplus\mathfrak{g}_{i+j+l-1}}\circ d\theta_{-k}(\xi_1-\xi_2,\eta)\\
=\ & pr_{\mathfrak{g}_{-k}\oplus\cdots\oplus\mathfrak{g}_{i+j+l-1}}\big((\xi_1-\xi_2)(\underbrace{\theta_{-k}(\eta)}_{=0})-\eta(\underbrace{\theta_{-k}(\xi_1-\xi_2)}_{=0})-\underbrace{\theta_{-k}([\xi_1-\xi_2,\eta])}_{=0}\big)\\
=\ & 0.
\end{aligned}$$

Thus in this case the torsion is well defined.

second case $i+l\geq 0$

Now it holds $\xi_1-\xi_2\in T_u^{i+l}E\subset T_u^0E$. Hence we have an element $A\in\mathfrak{g}^{i+l}$ such that its fundamental vector field at the point u is equal to $\xi_1-\xi_2$. So with the lemma above we can write for $i+j\geq -k$

$$\begin{aligned}
d\theta_{i+j}(\xi_1-\xi_2,\eta) &= d\theta_{i+j}(\widetilde{A},\eta)\\
&= -pr_{\mathfrak{g}_{i+j}\oplus\cdots\oplus\mathfrak{g}_{i+j+l-1}}\underbrace{[A,\theta_{i+j}(\eta)]_{\mathfrak{g}}}_{\in\mathfrak{g}^{i+j+l}}\\
&= 0
\end{aligned}$$

and in the same way for $i+j<-k$

$$\begin{aligned}
pr_{\mathfrak{g}_{-k}\oplus\cdots\oplus\mathfrak{g}_{i+j+l-1}}\circ d\theta_{-k}(\xi_1-\xi_2,\eta) &= pr_{\mathfrak{g}_{-k}\oplus\cdots\oplus\mathfrak{g}_{i+j+l-1}}\circ d\theta_{-k}(\widetilde{A},\eta)\\
&= -pr_{\mathfrak{g}_{-k}\oplus\cdots\oplus\mathfrak{g}_{i+j+l-1}}\underbrace{[A,\theta_{-k}(\eta)]_{\mathfrak{g}}}_{\in\mathfrak{g}^{i+j+l}}\\
&= 0.
\end{aligned}$$

Consequently the torsion of a P-frame bundle $(E,p,M;\theta)$ of degree l is well defined. The homogeneous components of the torsion are denoted by t_θ^j, $j=0,\dots,l-1$.

Definition 3.15 *A P-frame bundle $(E,p,M;\theta)$ of degree l is called harmonic if and only if for all $j=1,\dots,l-1$ we have $\partial^*\circ t_\theta^j=0$.*

Remark 3.3 *According to the structure equations we have for the underlying frame form of length one*

$$pr_{\mathfrak{g}_{i+j}}\circ d\theta_{i+j}(\xi,\eta)=-pr_{\mathfrak{g}_{i+j}}[\theta_i(\xi),\theta_j(\eta)]_{\mathfrak{g}}=-[X,Y]_{\mathfrak{g}}.$$

Thus the homogeneous component t_θ^0 of the torsion is already completely determined by the structure equations.

3.5.4 The Enlarged Bundle $\hat{E}$

Given $(E,p,M;\theta)$ a harmonic P-frame bundle of degree l we set

$$\hat{E}:=\left\{\varphi=(\varphi_{-k},\dots,\varphi_{-1})\ \middle|\ \begin{array}{l}\varphi_i\in L(T^iE,\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l})\text{ with}\\ pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l-1}}\circ\varphi_i(u)=\theta_i(u),\varphi_i|_{T_u^{i+1}E}=\theta_{i+1}(u),\\ \varphi_{-1}(\widetilde{A})=A\text{ for all }A\in\mathfrak{g}_0\oplus\cdots\oplus\mathfrak{g}_{l-1}\end{array}\right\}.$$

Please note that for elements $A\in\mathfrak{g}^0$ the fundamental vector field $\widetilde{A}$ is a section in $T_u^{i+1}E$ for $i=-k,\dots,-1$. Thus we have

$$\varphi_i(\widetilde{A})=\theta_{i+1}(u)(\widetilde{A})=\begin{cases}0, & i+l<0\\ pr_{\mathfrak{g}_0\oplus\cdots\oplus\mathfrak{g}_{i+l}}(A), & i+l\geq 0\end{cases}.$$

The obvious projection is denoted by $\pi:\hat{E}\longrightarrow E$.

Proposition 3.4 *$\pi : \hat{E} \longrightarrow E$ is a locally trivial bundle. Each fibre is an affine space with modeling vector space the space of all linear maps from $\mathfrak{g}_-$ to $\mathfrak{g}$ which are homogeneous of degree l,*

$$L_l(\mathfrak{g}_-, \mathfrak{g}) = \{\psi : \mathfrak{g}_- \longrightarrow \mathfrak{g} \text{ linear, homogeneous of degree } l\}.$$

Proof: Let φ and $\widetilde{\varphi}$ be two elements of $\hat{E}$ with the same projection, $\pi(\varphi) = \pi(\widetilde{\varphi}) = u$. So with $pr_{\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l-1}} \circ \varphi_i(u) = pr_{\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l-1}} \circ \widetilde{\varphi}_i(u) = \theta_i(u)$ we get $\widetilde{\varphi}_i - \varphi_i : T^i_u E \longrightarrow \mathfrak{g}_{i+l}$. Furthermore with $\varphi_i|_{T^{i+1}_u} = \widetilde{\varphi}_i|_{T^{i+1}_u} = \theta_{i+1}(u)$ we obtain $(\widetilde{\varphi}_i - \varphi_i)|_{T^{i+1}_u E} \equiv 0$ and we have a linear map

$$\widetilde{\varphi}_i - \varphi_i : T^i_u E/_{T^{i+1}_u E} \longrightarrow \mathfrak{g}_{i+l}.$$

Denote with Θ the frame form of length one which underlies θ. Since Θ_i induces an isomorphism $T^i_u E/_{T^{i+1}_u E} \xrightarrow{\sim} \mathfrak{g}_i$ there is a uniquely defined map

$$\psi_i : \mathfrak{g}_i \longrightarrow \mathfrak{g}_{i+l} \text{ such that } (\widetilde{\varphi}_i - \varphi_i)(\xi) = \psi_i \circ \Theta_i(\xi) \text{ for all } \xi \in T^i_u E.$$

This gives the affine structure of the fibres of $\pi : \hat{E} \longrightarrow E$ as the space of all linear maps $\psi = (\psi_{-k}, \ldots, \psi_{-1}) : \mathfrak{g}_- \longrightarrow \mathfrak{g}$ which are homogeneous of degree l.
To prove local triviality we construct local sections. Take an open subset $U \subset M$ such that all bundles $T^i M$, $i = -k, \ldots, -1$ and E are trivial over U. That means we can describe $TM|_U = U \times \mathfrak{g}_-$ as a filtered vector bundle and $E|_U = U \times P/_{P^l_+}$. Thus the tangent bundle of E can locally be written as

$$\begin{aligned} TE|_U &= TM|_U \times T(P/_{P^l_+}) \\ &= U \times \mathfrak{g}_- \times T(P/_{P^l_+}) \end{aligned}.$$

For $i = -k, \ldots, 0$ we have the projections

$$\begin{aligned} \pi^i : TE|_U &\longrightarrow T^i E|_U \\ (x, X_{-k} \oplus \cdots \oplus X_{-1}, A) &\mapsto \begin{cases} (x, X_i \oplus \cdots \oplus X_{-1}, A), & i = -k, \ldots, -1 \\ (x, 0, A), & i = 0 \end{cases} \end{aligned}.$$

With this we define

$$\begin{aligned} \hat{\theta}_i := \theta_{i+1} \circ \pi^{i+1} : T^i E &\longrightarrow \mathfrak{g}_{i+1} \oplus \cdots \oplus \mathfrak{g}_{i+l} \quad i = -k, \ldots, -2 \\ \hat{\theta}_{-1} := \theta_0 \circ \pi^0 : T^{-1} E &\longrightarrow \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{l-1} \end{aligned}$$

$$\text{where} \quad \begin{aligned} \theta_0 : T^0 E &\longrightarrow \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{l-1} \\ \widetilde{A} &\mapsto pr_{\mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{l-1}} A \end{aligned}$$

is the inverse of the fundamental vector field mapping.

Then we obtain a local section by setting

$$\begin{aligned} \varphi_i := \theta_i \oplus pr_{\mathfrak{g}_{i+l}} \hat{\theta}_i : T^i E \longrightarrow \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l}, \text{ for } i = -k, \ldots, -1 \\ \varphi = (\varphi_{-k}, \ldots, \varphi_{-1}) : E|_U \longrightarrow \hat{E}|_U. \end{aligned}$$

And φ_u is truly an element of $\hat{E}_u$ since

- according to the definition φ_i is an element of $L(T^i E, \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l})$ and the first l components coincide with θ_i, $pr_{\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l-1}} \circ \varphi_i(u) = \theta_i(u)$,
- restriction to $T^{i+1}_u E$ gives

$$\begin{aligned} \varphi_i|_{T^{i+1}_u E} &= (\theta_i \oplus pr_{\mathfrak{g}_{i+l}} \circ \hat{\theta}_i)|_{T^{i+1}_u E} \\ &= pr_{\mathfrak{g}_{i+1} \oplus \cdots \oplus \mathfrak{g}_{i+l-1}} \circ \theta_{i+1}(u) \oplus pr_{\mathfrak{g}_{i+l}} \circ \theta_{i+1}(u)\ , \\ &= \theta_{i+1}(u) \end{aligned}$$

- for $A \in \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{l-1}$ we have

$$\begin{aligned}\varphi_{-1}(\widetilde{A}) &= \theta_{-1}(\widetilde{A}) \oplus pr_{\mathfrak{g}_{l-1}} \circ \hat{\theta}_{-1}(\widetilde{A}) \\ &= pr_{\mathfrak{g}_0\oplus\cdots\oplus\mathfrak{g}_{l-2}}(A) \oplus pr_{\mathfrak{g}_{l-1}}(A) \ . \\ &= A\end{aligned}$$

Thus $\pi : \hat{E} \longrightarrow E$ is a locally trivial bundle with fibres isomorphic to the space of all linear maps from $\mathfrak{g}_-$ to $\mathfrak{g}$ which are homogeneous of degree l.

□

Now we define the action of $P/_{P_+^{l+1}}$ on $\hat{E}$. Let b be an element of $P/_{P_+^{l+1}}$ and denote its class in $P/_{P_+^l}$ with b_0. For $\varphi = (\varphi_{-k}, \ldots, \varphi_{-1}) \in \hat{E}$, $\varphi_i : T_u^i E \longrightarrow \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l}$ we set

$$\begin{aligned}R_b\varphi_i : T^i_{R_{b_0}u}E &\longrightarrow \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l} \\ \xi &\mapsto pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l}} \circ Ad(b^{-1}) \circ \varphi_i(u) \circ dR_{b_0^{-1}}(\xi),\end{aligned}$$

$$R_b\varphi := (R_b\varphi_{-k}, \ldots, R_b\varphi_{-1}).$$

To prove that this is actually an element of $\hat{E}$ again recall that the subgroup P_+^{l+1} acts trivial on $\mathfrak{g}^i/_{\mathfrak{g}^{i+l}}$ (using this fact will be denoted by $*$) as we have seen in Section 3.2. With this we find

- the first l components of $R_b\varphi_i$ coincide with θ_i,

$$\begin{aligned}pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l-1}} \circ R_b\varphi_i(R_{b_0}u) &= pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l-1}} \circ Ad(b^{-1}) \circ \varphi_i(u) \circ dR_{b_0^{-1}} \\ &= pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l-1}} \circ Ad(b^{-1}) \circ (\theta_i(u) \oplus pr_{\mathfrak{g}_{i+l}} \circ \varphi_i(u)) \circ dR_{b_0^{-1}} \\ &= pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l-1}} \circ Ad(b^{-1}) \circ \theta_i(u) \circ \varphi_i(u)) \circ dR_{b_0^{-1}} \\ &\overset{*}{=} pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l-1}} \circ \underbrace{Ad(b_0^{-1}) \circ \theta_i(u) \circ \varphi_i(u)) \circ dR_{b_0^{-1}}}_{=\theta_i(R_{b_0}u)} \\ &= \theta_i(R_{b_0}u),\end{aligned}$$

- restriction to $T_u^{i+1}E$ gives θ_{i+1},

$$\begin{aligned}R_b\varphi_i|_{T^{i+1}_{R_{b_0}u}E} &= pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l}} \circ Ad(b^{-1}) \circ \varphi_i(u) dR_{b_0^{-1}}|_{T^{i+1}_{R_{b_0}u}E} \\ &= pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l}} \circ Ad(b^{-1}) \circ \theta_{i+1}(u) dR_{b_0^{-1}} \\ &\quad \text{projection to } \mathfrak{g}_i \text{ of } \theta_{i+1}(u) \text{ vanishes} \\ &\overset{*}{=} pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l}} \circ Ad(b_0^{-1}) \circ \theta_{i+1}(u) dR_{b_0^{-1}} \\ &= pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l}} \circ \theta_{i+1}(R_{b_0}u) \\ &= \theta_{i+1}(R_{b_0}u),\end{aligned}$$

- the generators of the fundamental vector fields are reproduced by $R_b\varphi_{-1}$, for all elements $A \in \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{l-1}$ it holds

$$\begin{aligned}(R_b\varphi_{-1})(\widetilde{A}) &= pr_{\mathfrak{g}_{-1}\oplus\cdots\oplus\mathfrak{g}_{l-1}} \circ Ad(b^{-1}) \circ \varphi_{-1}(u) \circ \underbrace{dR_{b_0^{-1}}\big(\widetilde{A}(R_{b_0}u)\big)}_{=\widetilde{Ad(b_0)A(u)}} \\ &= pr_{\mathfrak{g}_{-1}\oplus\cdots\oplus\mathfrak{g}_{l-1}} \circ Ad(b^{-1}) \circ \varphi_{-1}(u)\big(\widetilde{Ad(b_0)}A(u)\big) \\ &= pr_{\mathfrak{g}_{-1}\oplus\cdots\oplus\mathfrak{g}_{l-1}} \circ Ad(b^{-1}) \circ Ad(b_0)(A) \\ &\overset{*}{=} pr_{\mathfrak{g}_{-1}\oplus\cdots\oplus\mathfrak{g}_{l-1}} \circ Ad(b_0^{-1}) \circ Ad(b_0)(A) \\ &= A.\end{aligned}$$

Assume that we have a $\varphi \in \hat{E}$ and an element $b \in P/_{P_+^{l+1}}$ with $R_b\varphi = \varphi$. Since the action of $P/_{P_+^l}$ on E is free, we obtain $b_0 = id$ and therefore $b = \exp(A)$ for an $A \in \mathfrak{g}_l$. In Subsection 3.2 we calculated that $Ad(-\exp(A))\varphi_i(\xi) = \varphi_i(\xi) - ad(A)(\varphi_i(\xi))$. For every $X \in \mathfrak{g}_{-1}$ we can find a $\xi \in T_u^{-1}E$ such that $\varphi_{-1}(\xi) = X$ and we obtain

$$\begin{aligned}
X = \varphi_{-1}(\xi) &\overset{!}{=} \big(R_b\varphi_{-1}\big)(\xi) \\
&= pr_{\mathfrak{g}_{-1}\oplus\cdots\oplus\mathfrak{g}_{l-1}} \circ Ad(-\exp(A)) \circ \varphi_{-1}(\xi) \\
&= pr_{\mathfrak{g}_{-1}\oplus\cdots\oplus\mathfrak{g}_{l-1}}(X - [A, X]_{\mathfrak{g}}) \\
&= X - [A, X]_{\mathfrak{g}}.
\end{aligned}$$

Thus for all $X \in \mathfrak{g}_{-1}$ we get $[A, X]_{\mathfrak{g}} = 0$. According to Subsection 3.2 this implies $A = 0$. So we obtain the following lemma.

Lemma 3.11 *The action of $P/_{P_+^{l+1}}$ on $\hat{E}$ is free.*

Furthermore by definition the projection $\pi : \hat{E} \longrightarrow E$ is equivariant over the canonical projection $P/_{P_+^{l+1}} \longrightarrow P/_{P_+^l}$, that is to say $\pi \circ R_b = R_{b_0} \circ \pi$.

3.5.5 The Natural Analog of a Frame Form

Now the tangent bundle $T\hat{E}$ inherits the filtration from TE, $T^i\hat{E} := d\pi^{-1}(T^iE)$ which is stable under the action of $P/_{P_+^{l+1}}$. We define a natural analog of a frame form of length $l+1$ on $\hat{E}$ by setting for $\xi \in T_\varphi^i\hat{E}$

$$\hat{\theta}_i(\xi) := \varphi_i \circ d\pi(\xi).$$

Thus $\hat{\theta}_i$ is a smooth section in $L(T^i\hat{E}, \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l})$. $\hat{\theta}_i$ satisfies

- the kernel of $\hat{\theta}_i|_{T_\varphi^i\hat{E}}$ is $T_\varphi^{i+l+1}\hat{E}$ since

$$\begin{aligned}
ker\hat{\theta}_i|_{T_\varphi^i\hat{E}} &= ker(\varphi_i \circ d\pi)|_{T_\varphi^i\hat{E}} \\
&= d\pi_\varphi^{-1}\big(ker\varphi_i|_{T_u^iE}\big) \\
&\subset d\pi_\varphi^{-1}\big(ker\theta_i(u) \cap ker\theta_{i+1}(u)\big) \\
&= T_\varphi^{i+l+1}\hat{E} \\
\text{and} \quad \hat{\theta}_i(T_\varphi^{i+l+1}\hat{E}) &= \varphi_i(T_u^{i+l+1}E) \\
&= \theta_{i+1}(T_u^{i+1+l}E) \\
&= 0,
\end{aligned}$$

- with $\hat{\theta}_i|_{T^{i+1}\hat{E}} = \varphi_i \circ d\pi|_{T^{i+1}\hat{E}}$ and φ restricted to $T^{i+1}E$ being identical to θ_{i+1} we obtain

$$\begin{aligned}
pr_{\mathfrak{g}_i} \circ \hat{\theta}_i|_{T^{i+1}\hat{E}} &= pr_{\mathfrak{g}_i} \circ \underbrace{\theta_{i+1}}_{\text{values in } \mathfrak{g}_{i+1}\oplus\cdots\oplus\mathfrak{g}_{i+l}} \circ d\pi|_{T^{i+1}\hat{E}} \\
&= 0 \\
\text{and} \quad pr_{\mathfrak{g}_{i+1}\oplus\cdots\oplus\mathfrak{g}_{i+l}} \circ \hat{\theta}_i|_{T^{i+1}\hat{E}} &= pr_{\mathfrak{g}_{i+1}\oplus\cdots\oplus\mathfrak{g}_{i+l}} \circ \theta_{i+1} \circ d\pi|_{T^{i+1}\hat{E}} \\
&= pr_{\mathfrak{g}_{i+1}\oplus\cdots\oplus\mathfrak{g}_{i+l}} \circ \varphi_{i+1} \circ d\pi|_{T^{i+1}\hat{E}} \\
&= pr_{\mathfrak{g}_{i+1}\oplus\cdots\oplus\mathfrak{g}_{i+l}} \circ \hat{\theta}_{i+1}|_{T^{i+1}\hat{E}},
\end{aligned}$$

- the components of $\hat{\theta}$ are $P/_{P_+^{l+1}}$-equivariant, because for vectors $\xi \in T^i_\varphi \hat{E}$ and elements $b \in P/_{P_+^{l+1}}$ we have

$$
\begin{aligned}
(R_b^*\hat{\theta}_i)(\xi) &= (R_b\hat{\theta}_i) \circ dR_b(\xi) \\
&= (R_b\varphi_i) \circ d\pi \circ dR_b(\xi) \\
&= pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l}} \circ Ad(b^{-1}) \circ \varphi_i \circ \underbrace{dR_{b_0^{-1}} \circ d\pi \circ dR_b}_{=d\pi}(\xi) \\
&= pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l}} \circ Ad(b^{-1}) \circ \hat{\theta}_i(\xi),
\end{aligned}
$$

- for $X \in \mathfrak{g}^0/_{\mathfrak{g}^{l+1}}$ we have according to the properties of φ_i

$$
\begin{aligned}
\hat{\theta}_i\big(\widetilde{X}(\varphi)\big) &= \varphi_i \circ d\pi\big(\widetilde{X}(\varphi)\big) \\
&= \varphi_i\big(\widetilde{X}(u)\big) \\
&= \begin{cases} 0, & i+l<0 \\ pr_{\mathfrak{g}_0\oplus\cdots\oplus\mathfrak{g}_{i+l}}X, & i+l\geq 0 \end{cases}.
\end{aligned}
$$

Although $\hat{\theta}$ fulfills all defining properties of a frame form it is just an analog of a frame form due to the fact that $\hat{E} \longrightarrow M$ is not a P-frame bundle since the action of $P/_{P_+^{l+1}}$ is not necessarily transitive on the fibres. Thus we need to modify $\hat{E}$ to obtain a P-frame bundle of degree $l+1$. This will be achieved by defining the torsion of elements of $\hat{E}$. However before we can do that, we have to further explore $\hat{\theta}$.

3.5.6 The Torsion of Elements of the Enlarged Bundle $\hat{E}$

Now let σ be a local section of $\pi : \hat{E} \longrightarrow E$ with $\sigma(u) = \varphi$. Then $\sigma^*\hat{\theta}_i$ is a smooth section of $L(T^iE, \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l})$ and its first l components coincide with θ_i of the frame form θ of E since this property is fulfilled by φ_i and

$$
\begin{aligned}
(\sigma^*\hat{\theta}_i)_u(\xi) &= (\hat{\theta}_i)_{\sigma(u)}(d\sigma\xi) \\
&= \varphi_i \circ d\pi \circ d\sigma(\xi) \\
&= \varphi_i(\xi).
\end{aligned}
$$

Lemma 3.12 *For $X \in \mathfrak{g}^0/_{\mathfrak{g}^l}$, $\xi \in T_u^{i+1}E$ and σ a local section of $\pi : \hat{E} \longrightarrow E$ around $u \in E$ it holds*

$$
d(\sigma^*\hat{\theta}_i)\big(\widetilde{X}_u, \xi_u\big) = -pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l}} \circ \big[X, (\sigma_i)_u(\xi)\big]_{\mathfrak{g}}.
$$

Proof: For $X \in \mathfrak{g}^0/_{\mathfrak{g}^{l+1}}$ it is $d\sigma(\widetilde{X}) = \widetilde{X}^{\hat{E}} + \lambda$, where $\widetilde{X}^{\hat{E}}$ denotes the fundamental vector field generated by X in the bundel $\hat{E}$ and $\lambda \in Tv_{\sigma(u)}\hat{E}$ is a vertical vector field with respect to $\pi : \hat{E} \longrightarrow E$. Thus we have

$$
\begin{aligned}
d(\sigma^*\hat{\theta}_i)(\widetilde{X}_u, \xi_u) &= d\hat{\theta}\big(d\sigma(\widetilde{X}_u), d\sigma(\xi_u)\big) \\
&= d\hat{\theta}\big(\widetilde{X}^{\hat{E}}_{\sigma(u)}, d\sigma(\xi_u)\big) + d\hat{\theta}\big(\lambda_{\sigma(u)}, d\sigma(\xi_u)\big).
\end{aligned}
$$

Following the same argumentations as in Lemma 3.10 we obtain

$$
\begin{aligned}
d\hat{\theta}\big(\widetilde{X}^{\hat{E}}_{\sigma(u)}, d\sigma(\xi_u)\big) &= -pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l}}\big[X, (\hat{\theta}_i)_{\sigma(u)}(d\sigma\xi)\big]_{\mathfrak{g}} \\
&= -pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l}}\big[X, (\sigma_i)_u \circ d\pi \circ d\sigma(\xi)\big]_{\mathfrak{g}} \\
&= -pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l}}\big[X, (\sigma_i)_u(\xi)\big]_{\mathfrak{g}}.
\end{aligned}
$$

Please note that, with $\xi \in T_u^{i+1}E$ as requested, $(\sigma_i)_u(\xi)$ has no component in $\mathfrak{g}_i$.

With the fibres of $\pi : \hat{E} \longrightarrow E$ being affine spaces with modeling vector space $L_l(\mathfrak{g}_-, \mathfrak{g})$ we can interpret the vertical vector field λ to be given by a linear map form $\mathfrak{g}_-$ to $\mathfrak{g}$ of homogeneous degree l, $\lambda = \widetilde{\psi}$ with $\psi \in L_l(\mathfrak{g}_-, \mathfrak{g})$. Again we compute the differential of $\hat{\theta}_i$ with the help of the Lie derivative.

$$\begin{aligned}
d\hat{\theta}_i(\widetilde{\psi}, d\sigma\xi) &= d\big(\underbrace{\hat{\theta}_i(\widetilde{\psi})}_{=0}\big)(d\sigma\xi) + d\hat{\theta}_i(\widetilde{\psi}, d\sigma\xi) \\
&= (d \circ i_{\widetilde{\psi}} + i_{\widetilde{\psi}} \circ d)(\hat{\theta}_i)(d\sigma\xi) \\
&= L_{\widetilde{\psi}}\hat{\theta}_i(d\sigma\xi) \\
&= \tfrac{d}{dt}\Big((\phi_t^{\widetilde{\psi}})^*\big(\hat{\theta}_i(d\sigma\xi)\big)\Big)\Big|_{t=0} \\
&= \tfrac{d}{dt}\Big((\phi_t^{\widetilde{\psi}})^*\big(\varphi_i \circ d\pi \circ d\sigma(\xi)\big)\Big)\Big|_{t=0} \\
&= \tfrac{d}{dt}\Big((\phi_t^{\widetilde{\psi}})^*\big(\varphi_i(\xi)\big)\Big)\Big|_{t=0} \\
&= \tfrac{d}{dt}\big(\varphi_i(\xi) + t\psi(\Theta_i(\xi))\big)\big|_{t=0} \\
&= \psi\big(\underbrace{\Theta_i\ \underbrace{(\xi)}_{\in T_u^{i+1}E}}_{=0}\ \big) \\
&= 0
\end{aligned}$$

Thus we have as claimed

$$d(\sigma^*\hat{\theta}_i)\big(\widetilde{X}_u, \xi_u\big) = -pr_{\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l}} \circ \big[X, (\sigma_i)_u(\xi)\big]_{\mathfrak{g}}.$$

$\square$

Now we can define the torsion of an element $\varphi \in \hat{E}$. For $X \in \mathfrak{g}_i$, $Y \in \mathfrak{g}_j$ and $\xi \in T_u^i E$, $\eta \in T_u^j E$ with $\varphi_i(\xi) = X$ and $\varphi_j(\eta) = Y$ we set

$$\begin{aligned}
t_\varphi(X, Y) &:= \begin{cases} pr_{\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{i+j+l}} \circ d(\sigma^*\hat{\theta}_{-k})_u(\xi, \eta), & i+j < -k \\ d(\sigma^*\hat{\theta}_{i+j})_u(\xi, \eta), & i+j \geq -k \end{cases} \\
&\in \mathfrak{g}_{i+j} \oplus \cdots \oplus \mathfrak{g}_{i+j+l}.
\end{aligned}$$

This is well defined because

- Assume we have $\xi_1, \xi_2 \in T_u^i E$ with $\varphi_i(\xi_1) = \varphi_i(\xi_2) = X$. Then the difference $\xi_1 - \xi_2$ is an element of the kernel of φ_i which is $T_u^{i+1+l}E$ since

$$\begin{aligned}
ker(\varphi_i) &\subset ker(\theta_i(u)) \cap ker\theta_{i+1}(u) \\
&= T_u^{i+1+l}E \\
&\text{and } \ \varphi_i(T_u^{i+1+l}E) = \theta_{i+1}(T_u^{i+1+l}E) = 0.
\end{aligned}$$

  In the special case of $i = -1$ the choice of ξ is unique since the kernel of φ_{-1} is trivial, $ker(\varphi_{-1}) = T_u^l E = 0$. Now let i be smaller than -1. For $i + 1 + l < 0$ we obtain that the torsion of φ is well defined in the same way as it was proven for the torsion of θ earlier in this section. For $i+1+l \geq 0$ we have an $A \in \mathfrak{g}_{i+1+l}$ such that $\xi_1 - \xi_2 = \widetilde{A}(u)$.

With $i < -1$ we have $i+1+l < l$ and using Lemma 3.12 we get

$$\begin{array}{rcl} \text{for } j > -k & & d(\sigma^*\hat{\theta}_{i+j})(\xi_1 - \xi_2, \eta) \\ & = & -pr_{\mathfrak{g}_{i+j}\oplus\cdots\oplus\mathfrak{g}_{i+j+l}} \underbrace{ad(A)(Y)}_{\in\mathfrak{g}_{i+j+l+1}} \\ & = & 0, \\ & & \text{eventually replace } i+j \text{ by } -k, \end{array}$$

$$\begin{array}{rcl} \text{for } j = -k & & pr_{\mathfrak{g}_{-k}\oplus\cdots\oplus\mathfrak{g}_{i+j+l}} \circ d(\sigma^*\hat{\theta}_{-k})(\xi_1 - \xi_2, \eta) \\ & = & -pr_{\mathfrak{g}_{-k}\oplus\cdots\oplus\mathfrak{g}_{i-k+l}}\big(ad(A)(Y) + \text{terms in } \mathfrak{g}_{-k+l}\big) \\ & = & 0. \end{array}$$

- Assume we have two local sections $\sigma, \overline{\sigma} : E|_U \longrightarrow \hat{E}|_U$ with $\sigma(u) = \overline{\sigma}(u)$. Since the fibres of $\pi : \hat{E} \longrightarrow E$ have modeling vector space $L_l(\mathfrak{g}_-, \mathfrak{g})$ there is a linear map $\psi : E|_U \longrightarrow L_l(\mathfrak{g}_-, \mathfrak{g})$ which is homogeneous of degree l such that $\overline{\sigma}_i = \sigma_i + \psi_i \circ \Theta_i$. Here Θ denotes again the underlying frame form of length one. $\sigma(u) = \overline{\sigma}(u)$ gives of course $\psi_u \equiv 0$. Using this results in

$$\begin{array}{rcl} (\overline{\sigma}^*\hat{\theta}_i)(\xi) & = & \hat{\theta}_i\big(d\overline{\sigma}(\xi)\big) \\ & = & \overline{\sigma}_i(\xi) \\ & = & \sigma_i(\xi) + \psi_i \circ \Theta_i(\xi) \\ & = & \big(\sigma^*\hat{\theta}_i + \psi_i \circ \Theta_i\big)(\xi). \end{array}$$

So for the differential and vectors $\xi \in T^i_u E$, $\eta \in T^j_u E$ we get

$$\begin{array}{rcl} d\big(\overline{\sigma}^*\hat{\theta}_{i+j}\big)(\xi, \eta) & = & d\big(\sigma^*\hat{\theta}_{i+j}\big)(\xi, \eta) + d\big(\psi \circ \Theta_{i+j}\big)(\xi, \eta) \\ & = & d(\sigma^*\hat{\theta}_{i+j})(\xi, \eta) + \xi\big(\psi \circ \underbrace{\Theta_{i+j}(\eta)}_{\equiv 0}\big) - \eta\big(\psi \circ \underbrace{\Theta_{i+j}(\xi)}_{\equiv 0}\big) - \psi \circ \Theta_{i+j}\big([\xi, \eta]\big) \\ & = & d(\sigma^*\hat{\theta}_{i+j})(\xi, \eta) + \psi\Big(\quad \xi\underbrace{(\Theta_{i+j}(\eta))}_{\equiv 0} - \eta\underbrace{(\Theta_{i+j}(\xi))}_{\equiv 0} - \Theta_{i+j}\big([\xi, \eta]\big) \quad \Big) \\ & = & d(\sigma^*\hat{\theta}_{i+j})(\xi, \eta) + \psi \circ d\Theta_{i+j}(\xi, \eta). \end{array}$$

  Joined by the structure equation this gives

$$d\big(\overline{\sigma}^*\hat{\theta}_{i+j}\big)(\xi, \eta) = d(\sigma^*\hat{\theta}_{i+j})(\xi, \eta) - \psi\big([\Theta_i(\xi), \Theta_j(\eta)]_{\mathfrak{g}}\big).$$

  At the point $u \in E$ we have $\psi_u \equiv 0$ and therefore $d(\overline{\sigma}^*\hat{\theta}_{i+j})_u = d(\sigma^*\hat{\theta}_{i+j})_u$. So the definition is independent of the section chosen.

In the thoughts preceding Lemma 3.12 we noticed that $pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l-1}} \circ \sigma^*\hat{\theta}_i \equiv \theta_i$. Thus the homogeneous components of the torsion t_φ of degree less than l coincide with $t_\theta(u)$. Hence it is sufficient to focus on the homogeneous component of degree l.
Now we want to study in which way the torsion t_φ depends on the point considered. So let $\widetilde{\varphi}_i = \varphi + \psi \circ \Theta_i$ be another point in $\hat{E}$. For $X \in \mathfrak{g}_i$ we choose a vector $\xi \in T^i_u E$ with $\varphi_i(\xi) = X$. I.e. we also have $\Theta_i(\xi) = X$ and consequently $\widetilde{\varphi}_i(\xi) = X + \underbrace{\psi(X)}_{\in\mathfrak{g}_{i+l}}$. We set ξ' to be $\widetilde{\psi(X)}$ for $i + l > 0$ or else we choose an element in $T^{i+l}_u E$ such that $\varphi_{i+l}(\xi') = \psi(X)$. With this definition we obtain

$$\begin{array}{rcl} \varphi_i(\xi') & = & \theta_{i+l}(\xi') = \varphi_{i+l}(\xi') = \psi(X), \\ \Theta_i(\xi') & = & 0 \text{ and} \\ \widetilde{\varphi}_i(\xi - \xi') & = & \varphi(\xi - \xi') + \psi \circ \Theta_i(\xi - \xi') = X. \end{array}$$

Similarly we choose for $Y \in \mathfrak{g}_j$ vectors η and η' with

$$\begin{aligned}
\varphi_i(\eta) &= \Theta_i(\eta) = Y, \\
\varphi_i(\eta') &= \psi(Y), \\
\Theta_i(\eta') &= 0 \text{ and} \\
\widetilde{\varphi}_i(\eta - \eta') &= Y.
\end{aligned}$$

In order to compute the torsion $t_{\widetilde{\varphi}}(X, Y)$ let $\widetilde{\sigma}$ be a local section with $\widetilde{\sigma}(u) = \widetilde{\varphi}$. Eventually replace $i + j$ by $-k$ if $i + j < -k$. As above we compute

$$\begin{aligned}
t_{\widetilde{\varphi}}(X, Y) &= d\big(\widetilde{\sigma}^*\hat{\theta}_{i+j}\big)(\xi - \xi', \eta - \eta') \\
&= d\big(\sigma^*\hat{\theta}_{i+j}\big)(\xi - \xi', \eta - \eta') - \underbrace{\psi\big([\Theta_i(\xi - \xi'), \Theta_j(\eta - \eta')]_{\mathfrak{g}}\big)}_{=\psi\left([X,Y]_{\mathfrak{g}}\right)} \\
&\qquad \text{(for } i + j < -k \text{ we have } [X, Y]_{\mathfrak{g}} = 0) \\
&= \underbrace{d\big(\sigma^*\hat{\theta}_{i+j}\big)(\xi, \eta)}_{=t_{\varphi}(X,Y)} - d\big(\sigma^*\hat{\theta}_{i+j}\big)(\xi', \eta) - d\big(\sigma^*\hat{\theta}_{i+j}\big)(\xi, \eta') \\
&\qquad + d\big(\sigma^*\hat{\theta}_{i+j}\big)(\xi', \eta') - \psi\big([X, Y]_{\mathfrak{g}}\big).
\end{aligned}$$

The term second to last vanishes since

$$\begin{aligned}
(\sigma^*\hat{\theta}_{i+j})(\xi') &= \hat{\theta}_{i+j} \circ d\sigma(\xi') = \sigma_{i+j}(\underbrace{\xi'}_{\in T_u^{i+l}E}) = 0, \\
(\sigma^*\hat{\theta}_{i+j})(\eta') &= 0 \text{ analogously}, \\
[\xi', \eta'] &\in \begin{cases} T_u^{i+j+2l}E, & \text{according to the thoughts succeeding} \\ & \text{definition 3.13 for } i + l < 0, j + l < 0 \\ T_u^{i+l}E, & \text{if } j + l \geq 0, i + l < 0 \\ T_u^{j+l}E, & \text{if } i + l \geq 0, j + l < 0 \\ T_u^{i+j+2l}E, & \text{if } i + l \geq 0, j + l \geq 0 \end{cases} \qquad \text{and} \\
(\sigma^*\hat{\theta}_{i+j})([\xi', \eta']) &= 0 \text{ since the highest component of } \hat{\theta}_{i+j} \text{ is } \mathfrak{g}_{i+j+l}.
\end{aligned}$$

Let us check the term $d(\sigma^*\hat{\theta}_{i+j})(\xi', \eta)$ now. For $i + l \geq 0$ the vector field $\xi' = \widetilde{\psi(X)}$ is fundamental. So according to Lemma 3.12 we have

$$\begin{aligned}
d(\sigma^*\hat{\theta}_{i+j})\big(\widetilde{\psi(X)}, \eta\big) &= -pr_{\mathfrak{g}_{i+j} \oplus \cdots \oplus \mathfrak{g}_{i+j+l}}\big[\underbrace{\psi(X)}_{\in \mathfrak{g}_{i+l}}, \underbrace{\sigma_{i+j}(\eta)}_{=Y \in \mathfrak{g}_j}\big]_{\mathfrak{g}} \\
&= \big[\psi(X), Y\big]_{\mathfrak{g}} \\
&\in \mathfrak{g}_{i+j+l}.
\end{aligned}$$

In the case of $i + l < 0$ we have $i + j + l < i + l$ and therefore with $\xi' \in T_u^{i+l}E$ the term $\sigma^*\hat{\theta}_{i+j}(\xi')$ vanishes. We also have $i + j + l < j$ and thus $\sigma^*\hat{\theta}_{i+j}(\eta) = 0$. So we have for the differential

$$\begin{aligned}
d(\sigma^*\hat{\theta}_{i+j})(\xi', \eta) &= \xi'\big(\underbrace{\sigma^*\hat{\theta}_{i+j}(\eta)}_{\equiv 0}\big) - \eta\big(\underbrace{\sigma^*\hat{\theta}_{i+j}(\xi')}_{\equiv 0}\big) - \sigma^*\hat{\theta}_{i+j}\big([\xi', \eta]\big) \\
&= -\sigma_{i+j}(\underbrace{[\xi', \eta]}_{\in T_u^{i+j+l}E})_u \\
&= -\theta_{i+j+1}\big([\xi'\eta]\big) \\
&= -pr_{\mathfrak{g}|i+j+l}\theta_{i+j+l}\big([\xi'\eta]\big) \\
&= -\Theta_{i+j}\big([\xi'\eta]\big) \\
&= d\Theta_{i+j}(\xi', \eta) && \text{since } \Theta_{i+j+l}(\xi') = 0 \text{ and } \Theta_{i+j+l}(\eta) = 0 \\
&= -\big[\Theta_{i+l}(\xi'), \Theta_j(\eta)\big]_{\mathfrak{g}} && \text{according to the structure equations} \\
&= -[\psi(X), Y]_{\mathfrak{g}}.
\end{aligned}$$

Consequently we obtain for the torsion

$$\begin{aligned} t_{\widetilde{\varphi}}(X,Y) &= t_\varphi(X,Y) + [\psi(X),Y]_{\mathfrak{g}} + [X,\psi(Y)]_{\mathfrak{g}} - \psi\big([X,Y]_{\mathfrak{g}}\big) \\ &= (t_\varphi + \partial\psi)(X,Y). \end{aligned}$$

3.5.7 The P-Frame Bundle of degree $l+1$

As we have seen in Section 3.3 the differential $\partial : L_l(\mathfrak{g}_- \wedge \mathfrak{g}_-, \mathfrak{g}) \longrightarrow L_l(\mathfrak{g}_-, \mathfrak{g})$ and the codifferential $\partial^* : L_l(\mathfrak{g}_-, \mathfrak{g}) \longrightarrow L_l(\mathfrak{g}_- \wedge \mathfrak{g}_-, \mathfrak{g})$ are adjoint to each other with respect to a certain metric. Hence we have especially $L_l(\mathfrak{g}_-, \mathfrak{g}) = Im\partial \oplus ker\partial^*$ and for every $\varphi \in \hat{E}$ there is a linear map $\psi \in L_l(\mathfrak{g}_-, \mathfrak{g})$ such that $t^l_{\varphi} + \partial\psi = t^l_{\varphi+\psi\circ\Theta}$ is an element of the kernel of the codifferential. Furthermore the space $\widetilde{E}_u := \{\varphi \in \hat{E}_u \mid \partial^*(t^l_\varphi) = 0\}$ has modeling vector space $ker\partial \subset L_l(\mathfrak{g}_-, \mathfrak{g})$.

Proposition 3.5 *Let $(E,p,M;\theta)$ be a harmonic P-frame bundle of degree l. The subbundle $\widetilde{E} := \{\varphi \in \hat{E} \mid \partial^*(t^l_\varphi) = 0\} \subset \hat{E}$ is closed under the right action of $P/_{P_+^{l+1}}$.*

Proof: Let φ be an element of $\widetilde{E}$ and $b \in P/_{P_+^{l+1}}$. We need to prove that $R_b\varphi \in \widetilde{E}$, i.e. we need to compute the torsion of this element. In order to keep the notation manageable we will ignore the case of $i+j < -k$. So for this case one would have to insert $-k$ for $i+j$ in the computations. Again denote with b_0 the class of b in $P/_{P_+^l}$. If σ is a local section of $\pi : \hat{E} \longrightarrow E$ with $\sigma \circ \pi(\varphi) = \sigma(u) = \varphi$ then $\overline{\sigma} := R_b \circ \sigma \circ R_{b_0^{-1}}$ is a local section with $\overline{\sigma} \circ \pi(R_b\varphi) = \overline{\sigma}(R_{b_0}u) = R_b\varphi$. We have

$$\begin{aligned} \overline{\sigma}^*\hat{\theta}_i &= (R_b \circ \sigma \circ R_{b_0^{-1}})^*\hat{\theta}_i \\ &= R^*_{b_0^{-1}}\big(\sigma^*(R^*_b\hat{\theta}_i)\big) \\ &= R^*_{b_0^{-1}}\big(\sigma^*(pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l}} \circ Ad(b^{-1}) \circ \hat{\theta}_i)\big) \\ &= pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l}} \circ Ad(b^{-1}) \circ R^*_{b_0^{-1}}\big(\sigma^*\hat{\theta}_i\big). \end{aligned}$$

So for the differential we get

$$d(\overline{\sigma}^*\hat{\theta}_{i+j})(\xi,\eta) = pr_{\mathfrak{g}_{i+j}\oplus\cdots\oplus\mathfrak{g}_{i+j+l}} \circ Ad(b^{-1}) \circ d(\sigma^*\hat{\theta}_{i+j})\big(dR_{b_0^{-1}}\xi, dR_{b_0^{-1}}\eta\big).$$

Here $\xi \in T^i_{R_{b_0}u}E$ needs to fulfill

$$(R_b\varphi_i)(\xi) = pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l}} \circ Ad(b^{-1}) \circ \varphi_i \circ dR_{b_0^{-1}}(\xi) \stackrel{!}{=} X \in \mathfrak{g}_i.$$

And analogously $\eta \in T^j_{R_{b_0}u}E$ fulfills $(R_b\varphi_j)(\eta) \stackrel{!}{=} Y \in \mathfrak{g}_j$. We split $\xi = dR_{b_0}(\xi' + \xi'')$ such that

$$\begin{aligned} \varphi_i(\xi') &\stackrel{!}{=} pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{-1}} \circ Ad(b)X =: Ad_-(b)X \text{ and} \\ \varphi_i(\xi'') &\stackrel{!}{=} Ad(b)X - \varphi_i(\xi') =: Ad_+(b)X. \end{aligned}$$

With this choice we have $(R_b\varphi_i)(\xi) = X \in \mathfrak{g}_i$. Analogously we splitt $\eta = dR_{b_0}(\eta' + \eta'')$ with

$$\begin{aligned} \varphi_i(\eta') &\stackrel{!}{=} Ad_-(b)Y \text{ and} \\ \varphi_i(\eta'') &\stackrel{!}{=} Ad_+(b)Y. \end{aligned}$$

Now we can use Lemma 3.12.

$$\begin{aligned} d(\sigma^*\hat{\theta}_{i+j})\big(dR_{b_0^{-1}}\xi, dR_{b_0^{-1}}\eta\big) &= d(\sigma^*\hat{\theta}_{i+j})\big(\xi'+\xi'', \eta'+\eta''\big) \\ &= d(\sigma^*\hat{\theta}_{i+j})\big(\xi', \eta'\big) \\ &\quad + \underbrace{d(\sigma^*\hat{\theta}_{i+j})\big(\xi'', \eta'+\eta''\big)}_{\stackrel{mod\ \mathfrak{g}^{i+j+l+1}}{\equiv} -[Ad_+(b)X, Ad(b)Y]_{\mathfrak{g}}} + \underbrace{d(\sigma^*\hat{\theta}_{i+j})\big(\xi', \eta''\big)}_{\stackrel{mod\ \mathfrak{g}^{i+j+l+1}}{\equiv} -[Ad_-(b)X, Ad_+(b)Y]_{\mathfrak{g}}} \\ &\equiv d(\sigma^*\hat{\theta}_{i+j})\big(\xi', \eta'\big) \\ &\quad -[Ad_+(b)X, Ad(b)Y]_{\mathfrak{g}} - [Ad_-(b)X, Ad_+(b)Y]_{\mathfrak{g}}\ mod\ \mathfrak{g}^{i+j+l+1} \end{aligned}$$

In order to compute the first term split ξ' and η' in the following way:

$$\begin{aligned} \xi' &= \xi'_i + \cdots + \xi'_{-1} \quad \text{with} \quad \varphi_i(\xi'_\mu) = pr_{\mathfrak{g}_\mu} \circ Ad(b)X \text{ and} \\ \eta' &= \eta'_j + \cdots + \eta'_{-1} \quad \text{with} \quad \varphi_j(\eta'_\nu) = pr_{\mathfrak{g}_\nu} \circ Ad(b)Y. \end{aligned}$$

So computing modulo $\mathfrak{g}^{i+j+l+1}$ we solely need to consider $\mu+\nu \leq i+j+l$ and by using the defining properties of φ and θ we obtain

$$\begin{aligned} d(\sigma^*\hat{\theta}_{i+j})(\xi'_\mu, \eta'_\nu) &= -(\sigma^*\hat{\theta}_{i+j})\big([\xi'_\mu, \eta'_\nu]\big) \\ &= -\varphi_{i+j}\big([\xi'_\mu, \eta'_\nu]\big) \\ &= -\theta_{\mu+\nu}\big([\xi'_\mu, \eta'_\nu]\big) \\ &= -\varphi_{\mu+\nu}\big([\xi'_\mu, \eta'_\nu]\big) \\ &= d(\sigma^*\hat{\theta}_{\mu+\nu})(\xi'_\mu, \eta'_\nu). \end{aligned}$$

According to the structure equations the homogeneous components of degree zero add up to

$$\sum_{\mu,\nu} -[\Theta_\mu(\xi'_\mu), \Theta_\nu(\eta'_\nu)] = \big[Ad_-(b)X, Ad_-(b)Y\big]_{\mathfrak{g}}.$$

Thus we finally obtain for the differential defining the torsion

$$\begin{aligned} d(\overline{\sigma}^*\hat{\theta}_{i+j})(\xi, \eta) &= pr_{\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+j+l}} \circ Ad(b^{-1}) \circ d(\sigma^*\hat{\theta}_{i+j})(\xi' + \xi'', \eta' + \eta'') \\ &= pr_{\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+j+l}} \circ Ad(b^{-1}) \Big(-\big[Ad(b)X, Ad(b)Y\big]_{\mathfrak{g}} \\ &\qquad\qquad + t_\varphi^{\geq 1}\big(Ad_-(b)X, Ad_-(b)Y\big)\Big) \\ &= -[X,Y]_{\mathfrak{g}} + pr_{\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+j+l}} \circ Ad(b^{-1}) \circ t_\varphi^{\geq 1}\big(Ad_-(b)X, Ad_-(b)Y\big). \end{aligned}$$

Recall that the homogeneous components of the torsion t_φ^μ coincide with the torsion of the frame form θ for $\mu = 1, \ldots, l-1$. So with $(E, p, M; \theta)$ being harmonic and $\varphi \in \widetilde{E}$ we have $\partial^* t_\varphi^{\geq 1} \equiv 0$. Thus with the equivariancy of the codifferential ∂^* the statement claimed follows

$$\partial^* t^l_{R_b\varphi} = 0, \text{ i.e. } R_b\varphi \in \widetilde{E}.$$

□

Assume now that the first cohomology group of degree l vanishes, $H^1_l(\mathfrak{g}_-, \mathfrak{g}) = 0$.

- If $l > k$ we can find for each point $u \in E$ a unique $\varphi \in \hat{E}$ which has ∂^*-closed torsion, $\partial^* t_\varphi = 0$. This gives a smooth global section σ of $\pi : \hat{E} \longrightarrow E$ which is according to Proposition 3.5 $P/_{P^l_+} = P$-equivariant. As we have seen prior to Lemma 3.12 $\hat{\theta}$ satisfies all conditions of a frame form of length $l+1$. It also fulfills the structure equations since its underlying frame form of length one coincides with the one of θ. So by pulling back $\hat{\theta}$ along the global section of $\pi : \hat{E} \longrightarrow E$ we obtain a frame form of length $l+1$ on E and thus this upgrades E to be a P-frame bundle of degree $l+1$. This is actually harmonic since according to the definition the following torsions coincide $t_{\sigma^*\hat{\theta}}(u) = t_{\sigma(u)}$ and thus $\partial^* t^j_{\sigma^*\hat{\theta}}(u) = \partial^* t^j_{\sigma(u)} = 0$ for $j = 1, \ldots, l$.
- If $l \leq k$ we introduce the following notations

$$\begin{aligned} &\widetilde{E} := \big\{\varphi \in \hat{E} \bigm| \partial^*(t^l_\varphi) = 0\big\}, \\ &\widetilde{p} : \widetilde{E} \longrightarrow M \text{ the obvious projection and} \\ &\widetilde{\theta} \text{ the restriction of } \hat{\theta} \text{ to } \widetilde{E}. \end{aligned}$$

  By Proposition 3.5 and Lemma 3.11 we have a free right action of $P/_{P^{l+1}_+}$ on $\widetilde{E}$ which preserves the fibres of $\widetilde{p}$. To prove that this action is also transitive on each fibre take two arbitrary points φ and $\overline{\varphi}$ of the same fibre $\widetilde{E}_x$. Then $\pi(\varphi), \pi(\overline{\varphi}) \in E_x$ are elements of the same fibre of the $P/_{P_+^l}$-principal bundle $p : E \longrightarrow M$. Thus we have $b_0 \in P/_{P_+^l}$ with $R_{b_0}\pi(\varphi) = \pi(\overline{\varphi})$.

Using the canonical section

$$\begin{array}{rcl} s : P/_{P_+^l} & \longrightarrow & P/_{P_+^{l+1}} \\ g_0 \exp(X_1) \cdots \exp(X_{l-1}) \cdot P_+^l & \mapsto & g_0 \exp(X_1) \cdots \exp(X_{l-1}) \cdot P_+^{l+1} \end{array}$$

we obtain $\pi(R_{s(b_0)}\varphi) = R_{b_0} \circ \pi(\varphi) = \pi(\overline{\varphi})$. Thus according to the facts preceding Proposition 3.5 we have a $\psi \in Ker(\partial) \subset L_l(\mathfrak{g}_-, \mathfrak{g})$ such that

$$\overline{\varphi}_i(\xi) = \big(R_{s(b_0)}\varphi_i\big)(\xi) + \psi\big(\Theta_i(\xi)\big).$$

With the vanishing of the cohomology group, $H_l^1(\mathfrak{g}_-, \mathfrak{g}) = 0$, the map $\psi \in Ker(\partial)$ is also an element of the image of $\partial : \mathfrak{g}_l \longrightarrow L_l(\mathfrak{g}_-, \mathfrak{g})$, that is we have some $A \in \mathfrak{g}_l$ with $\psi = -ad(A)$. Using Lemma 3.3 and similar calculations as in the proof of Lemma 3.4 we obtain

$$\begin{array}{rl} & \big(R_{\exp(A)} \circ R_{s(b_0)} \circ \varphi_i\big)(\xi) \\ = & pr_{\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l}} \circ Ad(\exp(-A)) \circ \big(R_{s(b_0)}\varphi_i\big)(\xi) \\ \overset{3.3,\ 3.4}{=} & pr_{\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l}}\Big(\underbrace{(R_{s(b_0)}\varphi_i)(\xi)}_{\in \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l}} - ad(\underbrace{A}_{\in \mathfrak{g}_l}) \circ \underbrace{\big(R_{s(b_0)}\varphi_i\big)(\xi)}_{\in \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l}}\Big) \\ = & (R_{s(b_0)}\varphi_i)(\xi) - ad(A)\big(\underbrace{pr_{\mathfrak{g}_i} \circ R_{s(b_0)} \circ \varphi_i(\xi)}_{=\Theta_i(\xi)}\big) \\ = & (R_{s(b_0)}\varphi_i)(\xi) - ad(A)\big(\Theta_i(\xi)\big) \\ = & \overline{\varphi}_i(\xi). \end{array}$$

I.e. we have $\overline{\varphi} = R_{s(b_0)\exp(A)}\varphi$ with $s(b_0)\exp(A) \in P/_{P_+^{l+1}}$. Thus the action of $P/_{P_+^{l+1}}$ is transitive on each fibre of $\widetilde{E}$.

With $\pi : \hat{E} \longrightarrow E$ and $p : E \longrightarrow M$ being locally trivial bundles we also have local smooth sections for $\widetilde{p} : \widetilde{E} \longrightarrow M$. Thus $\widetilde{p} : \widetilde{E} \longrightarrow M$ is a $P/_{P_+^{l+1}}$-principal bundle endowed with the frame form $\widetilde{\theta}$ of length $l+1$ which satisfies the structure equations according to the construction. I.e. $(\widetilde{E}, \widetilde{p}, M; \widetilde{\theta})$ is a P-frame bundle of degree $l+1$.

It remains to discuss wether $\widetilde{\theta}$ is harmonic. As we have seen in the proof of Lemma 3.12 inserting a vector field λ which is vertical with respect to $\pi : \hat{E} \longrightarrow E$ causes the torsion $t_{\hat{\theta}}$ to vanish. Thus it is sufficient to consider the torsion $t_{\sigma^*\hat{\theta}}$. As in the case of $l > k$ we have $t_{\sigma^*\hat{\theta}}(u) = t_{\sigma(u)}$ and thus $\partial^* t^j_{\sigma^*\hat{\theta}}(u) = \partial^* t^j_{\sigma(u)} = 0$ for $j = 1, \ldots, l$. I.e. $(\widetilde{E}, \widetilde{p}, M; \widetilde{\theta})$ is a harmonic P-frame bundle of degree $l+1$.

So we have constructed a harmonic P-frame bundle of degree $l+1$ out of a harmonic P-frame bundle of degree l.

3.5.8 Uniqueness

To discuss uniqueness we need to identify the underlying P-frame bundle of degree l for any P-frame bundle of degree $l+1$.
Assume that $(\widetilde{E}, \widetilde{\theta})$ is any P-frame bundle of degree $l+1$. If $l > k$ we obtain a P-frame bundle of degree l by simply omitting the last component of each $\widetilde{\theta}_i$. In the case of $l \leq k$ we set $E := \widetilde{E}\Big/_{\left(P_+^l/_{P_+^{l+1}}\right)}$ and denote the projection with $\pi : \widetilde{E} \longrightarrow E$. The frame form

is defined by choosing for $\xi \in T^i_u E$ a point $\varphi \in \widetilde{E}_u$ and a vector $\widetilde{\xi} \in T_\varphi \widetilde{E}$ with $d\pi(\widetilde{\xi}) = \xi$. Then we set

$$\theta_i(\xi) := pr_{\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l-1}} \circ \widetilde{\theta}_i(\widetilde{\xi}).$$

We need to prove that this is well defined. Given $\widetilde{\xi} \in T_\varphi \widetilde{E}$ with $d\pi(\widetilde{\xi}) = 0$ we have an element $A \in \mathfrak{g}_l$ with $\widetilde{\xi} = \widetilde{A}_\varphi \in T^l_\varphi \widetilde{E} \subset ker \widetilde{\theta}_i$. Thus the definition of the frame form is independent of the lift $\widetilde{\xi}$ of the vector ξ. For two points $\varphi_1, \varphi_2 \in \widetilde{E}_u$ we have an element $\exp(A) \in P^l_+ / _{P^{l+1}_+}$, $A \in \mathfrak{g}_l$ such that $R_{\exp(A)}\varphi_1 = \varphi_2$. Then we get

$$\begin{aligned} pr_{\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l-1}} \circ (\widetilde{\theta}_i)_{R_{\exp(A)}\varphi}\big(dR_{\exp(A)}\widetilde{\xi}_1\big) &= pr_{\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l-1}} \circ (R^*_{\exp(A)}\widetilde{\theta}_i)_\varphi\big(\widetilde{\xi}_1\big) \\ &= pr_{\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l-1}} \circ Ad(\exp(-A)) \circ (\widetilde{\theta}_i)_\varphi\big(\widetilde{\xi}_1\big). \end{aligned}$$

However according to Lemma 3.3 this is just $pr_{\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l-1}} \circ (\widetilde{\theta}_i)_\varphi\big(\widetilde{\xi}_1\big)$. I.e. the definition of the frame form is also independent of the choice of $\varphi \in \widetilde{E}_u$.
Since the frame form θ was constructed using $\widetilde{\theta}$ it inherits directly the properties of satisfying the structure equations and being harmonic from $\widetilde{\theta}$. More precisely for $l > k$ both claims are trivial since θ_i was defined to be $\widetilde{\theta}$ with the last component omitted. For $l \leq k$ given a local section σ of $\pi : \widetilde{E} \longrightarrow E$ we obtain $pr_{\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l-1}} \circ \sigma^* \widetilde{\theta}_i = \theta_i$. So the underlying frame forms of length one, $\widetilde{\Theta}$ resp. Θ, locally satisfy $\sigma^*\widetilde{\Theta}_{-k} = \Theta_{-k}$ and so do $d(\sigma^*\widetilde{\Theta}_{-k})$ and $d\Theta_{-k}$, which implies the vanishing of the structure equation of degree $-k$ on E. Now extending $\widetilde{\Theta}_{-k+1}$ to a one form and pulling it back by σ gives an extension of Θ_{-k+1} with $d\Theta_{-k+1} = d(\sigma^*\widetilde{\Theta}_{-k+1})$ locally. And so on ... Thus with $\widetilde{\theta}$ satisfying the structure equations so does θ. Considering the torsion we get with $\theta_i = pr_{\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l-1}} \circ \sigma^* \widetilde{\theta}_i$ as above also

$$t_\theta(u)(X,Y) = pr_{\mathfrak{g}_{i+j} \oplus \cdots \oplus \mathfrak{g} i+j+l-1} d\widetilde{\theta}_{i+j}(d\sigma\xi, d\sigma\eta)$$

for $X \in \mathfrak{g}_i$, $Y \in \mathfrak{g}_j$ and $\xi \in T^i_u E$, $\eta \in T^j_u E$ with $\theta(\xi) = X$ and $\theta_j(\eta) = Y$ as usual. However according to the thoughts succeeding the proof of Lemma 3.10 this coincides with $pr_{\mathfrak{g}_{i+j} \oplus \cdots \oplus \mathfrak{g} i+j+l-1} d\widetilde{\theta}_{i+j}(\widetilde{\xi}, \widetilde{\eta})$ for $\widetilde{\xi} \in T^i_{\sigma(u)}\widetilde{E}$ and $\widetilde{\eta} \in T^j_{\sigma(u)}\widetilde{E}$ with $\widetilde{\theta}_i(\widetilde{\xi}) = X$ and $\widetilde{\theta}_j(\widetilde{\eta}) = Y$. Thus with $\widetilde{\theta}$ being harmonic so is θ.
I.e. we have identified the underlying harmonic P-frame bundle of degree l of a harmonic P-frame bundle of degree $l+1$. Now we can address the question of uniqueness. Assume that $(\widetilde{E}, \widetilde{\theta})$ is any P-frame bundle of degree $l+1$ with (E, θ) as underlying P-frame bundle of degree l, that is to say we have especially a smooth projection $\pi : \widetilde{E} \longrightarrow E$ which is equivariant over the canonical projection $P/_{P^{l+1}_+} \longrightarrow P/_{P^l_+}$, $\pi \circ R_{p \cdot P^{l+1}_+} = R_{p \cdot P^l_+} \circ \pi$. At first we will define a smooth fibre bundle homomorphism $f : \widetilde{E} \longrightarrow \hat{E}$.
Recall the definition of $\hat{E}$:

$$\hat{E} := \left\{ \varphi = (\varphi_{-k}, \ldots, \varphi_{-1}) \;\middle|\; \begin{array}{l} \varphi_i \in L(T^i E, \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l}) \text{ with} \\ pr_{\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l-1}} \circ \varphi_i(u) = \theta_i(u), \varphi_i|_{T^{i+1}_u E} = \theta_{i+1}(u), \\ \varphi_{-1}(\widetilde{A}) = A \text{ for all } A \in \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{l-1} \end{array} \right\}.$$

Given $\widetilde{u} \in \widetilde{E}$ with $\pi(\widetilde{u}) = u$ we define $f(\widetilde{u}) = \big(f(\widetilde{u})_{-k}, \ldots, f(\widetilde{u})_{-1}\big)$ via

$$f(\widetilde{u})_i(\xi) := (\widetilde{\theta}_i)_{\widetilde{u}}(\widetilde{\xi}) \text{ where we choose for } \xi \in T^i_u E \text{ a vector } \widetilde{\xi} \in T^i_{\widetilde{u}}\widetilde{E} \text{ with } d\pi(\widetilde{\xi}) = \xi.$$

To see that this is well defined, let $\widetilde{\xi} \in T^i_{\widetilde{u}}\widetilde{E}$ be a vector with $d\pi(\widetilde{\xi}) = 0$. Thus $\widetilde{\xi} = \widetilde{A}$ is the fundamental vector field of an element $A \in \mathfrak{g}_l$. However with the defining properties for frame forms we have $(\widetilde{\theta}_i)_{\widetilde{u}}(\widetilde{A}) = pr_{\mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{i+l}}(A) = 0$. Furthermore $f(\widetilde{u})$ is actually an element of $\hat{E}$ since $\widetilde{\theta}$ is a frame form of length $l+1$ (denoted in the following argumentation by $*$) and the underlying P-frame bundle of degree l is (E,θ) (denoted by $**$).

$$\begin{array}{rcl} f(\widetilde{u})_i & = & (\widetilde{\theta}_i)_{\widetilde{u}} \\ & \overset{*}{\in} & L(T^i_u E, \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{i+l}) \end{array}$$

$$\begin{array}{rcl} pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l-1}} \circ f(\widetilde{u})_i & = & pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l-1}} \circ (\widetilde{\theta}_i)_{\widetilde{u}} \\ & \overset{**}{=} & \theta_i(u) \end{array}$$

$$\begin{array}{rcl} f(\widetilde{u})_i|_{T^{i+1}_u E} & = & (\widetilde{\theta}_i)_{\widetilde{u}}|_{T^{i+1}_u E} \\ & \overset{*}{=} & pr_{\mathfrak{g}_{i+1}\oplus\cdots\oplus\mathfrak{g}_{i+l}} \circ (\widetilde{\theta}_{i+1})_{\widetilde{u}} \\ & \overset{**}{=} & \theta_{i+1}(u) \end{array}$$

For $A \in \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{l-1}$ we have

$$\begin{array}{rcl} f(\widetilde{u})_{-1}(\widetilde{A}) & = & (\widetilde{\theta}_{-1})_{\widetilde{u}}(\widetilde{A}) \\ & \overset{*}{=} & pr_{\mathfrak{g}_0\oplus\cdots\oplus\mathfrak{g}_{l-1}}(A) \\ & = & A. \end{array}$$

Thus we get $f(\widetilde{u}) \in \hat{E}$ and $f : \widetilde{E} \longrightarrow \hat{E}$ is a smooth fibre bundle homomorphism. Computing $f(R_p\widetilde{u})$ for $p \in P/_{P^{l+1}_+}$ we see that f is $P/_{P^{l+1}_+}$-equivariant. For $\xi \in T^i_u E$ and $\widetilde{\xi} \in T^i_{\widetilde{u}}\widetilde{E}$ a lift of ξ we have $d\pi(dR_p\widetilde{\xi}) = dR_{p_0}\xi$, where p_0 denotes the class of p in $P/_{P^l_+}$. With this and using the $P/_{P^{l+1}_+}$-equivariancy of the frame form $\widetilde{\theta}$ and the definition of the $P/_{P^{l+1}_+}$-action on $\hat{E}$ we have

$$\begin{array}{rcl} f(R_p\widetilde{u})_i(dR_{p_0}\xi) & = & (\widetilde{\theta}_i)_{R_p\widetilde{u}}\big(dR_p\widetilde{\xi}\big) \\ & = & pr_{\mathfrak{g}_i\oplus\cdots\oplus\mathfrak{g}_{i+l}} \circ Ad(p^{-1}) \circ (\widetilde{\theta}_i)_{\widetilde{u}}(\widetilde{\xi}) \\ & = & \big(R_p(\widetilde{\theta}_i)_{\widetilde{u}}\big)(dR_{p_0}\widetilde{\xi}) \\ & = & \big(R_p f(\widetilde{u})\big)(dR_{p_0}\xi). \end{array}$$

The pullback of $\hat{\theta}$ with f gives the frame form $\widetilde{\theta}$ since according to the definition of $\hat{\theta}$ we have

$$\begin{array}{rcl} (f^*\hat{\theta}_i)_{\widetilde{u}}(\widetilde{\xi}) & = & (\hat{\theta}_i)_{f(\widetilde{u})}\big(df\widetilde{\xi}\big) \\ & \overset{def\ \hat{\theta}}{=} & d(\widetilde{u})_i \circ d\hat{\pi} \circ df(\widetilde{\xi}) \\ & = & f(\widetilde{u})_i \circ d\pi(\widetilde{\xi}) \\ & = & (\widetilde{\theta}_i)_{\widetilde{u}}(\widetilde{\xi}). \end{array}$$

With arguments similar to the ones verifying that $\widetilde{\theta}$ was harmonic if θ was on the previous page we find that with the definition $f(\widetilde{u})(\xi) = (\widetilde{\theta}_i)_{\widetilde{u}}(\widetilde{\xi})$ the torsions coincide, $t_{\widetilde{\theta}(\widetilde{u})} = t_{f(\widetilde{u})}$. Taking these facts together we obtain that

$$\begin{array}{ccc} f : \widetilde{E} & \longrightarrow & Im(f) \subset \hat{E} \\ \pi \searrow & & \swarrow \hat{\pi} \\ & E & \end{array}$$

is actually an isomorphism of P-frame bundles of degree $l+1$ and the image of f contains exactly the torsion free elements of $\hat{E}$ as in the prolongation construction.
Thus the constructed prolongation $(\widetilde{E}, \widetilde{p}, M; \widetilde{\theta})$ of the P-frame bundle $(E, p, M; \theta)$ of degree l is unique up to isomorphism and the following proposition is proven.

Proposition 3.6 *Let $(E, p, M; \theta)$ be a harmonic P-frame bundle of degree l, and suppose that the cohomology group $H^1_l(\mathfrak{g}_-, \mathfrak{g})$ vanishes. Then there is an (up to isomorphism) unique harmonic P-frame bundle $(\widetilde{E}, \widetilde{p}, M; \widetilde{\theta})$ of degree $l+1$ whose underlying P-frame bundle of degree l is isomorphic to $(E, p, M; \theta)$.*

Iterated application gives

Corollary 3.1 *Suppose that G is a semisimple Lie group whose Lie algebra $\mathfrak{g}$ is endowed with a $|k|$-grading, such that all cohomology groups $H^1_l(\mathfrak{g}_-, \mathfrak{g})$ with $l > 0$ are trivial. Furthermore let M be a smooth manifold endowed with a filtration of its tangent bundle $T^{-k}M = TM \supset T^{-k+1}M \supset \cdots \supset T^{-1}M$ by vector subbundles, such that for each $i = -k, \ldots, -1$ the rank of T^iM equals the dimension of $\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{-1}$.*
Then there is a bijective correspondence between isomorphism classes of reductions to the structure group G_0 of the associated graded vector bundle to the tangent bundle, which satisfy the structure equations, and isomorphism classes of P-principal bundles over M endowed with Cartan connections with ∂^*-closed curvature and $(\Omega^\omega)^l \equiv 0$ for all $l \le 0$.

Remark 3.4 *With $\omega = \theta_{-k}$ being the $-k$-component of the frame form of length $2k+1$ the property of θ being harmonic (i.e. $\partial^* \circ t^l_\theta = 0$ for $l = 1, \ldots, 2k$) is equivalent to $\partial^*(\Omega^\omega)^l \equiv 0$ for $l > 0$, since with $X_i \in \mathfrak{g}_i$ and $X_j \in \mathfrak{g}_j$ it holds*

$$\begin{aligned}
\Omega^\omega\big(\omega^{-1}(X_i), \omega^{-1}(X_j)\big) &= d\omega\big(\omega^{-1}(X_i), \omega^{-1}(X_j)\big) + \tfrac{1}{2}[\omega, \omega]^\wedge(\omega^{-1}(X_i), \omega^{-1}(X_j)) \\
&= d\omega\big(\omega^{-1}(X_i), \omega^{-1}(X_j)\big) + [X_i, X_j]_\mathfrak{g} \\
&= t_\theta(X_i, X_j) + \underbrace{[X_i, X_j]_\mathfrak{g}}_{\in \mathfrak{g}_{i+j}}.
\end{aligned}$$

The frame form θ with the underlying frame form θ^1 of length one satisfies the structure equations if and only if the homogeneous component of degree zero of the curvature of the Cartan connection vanishes since

$$\begin{aligned}
0 &\overset{!}{=} d\theta^1_{-k}\big(\omega^{-1}(X_i), \omega^{-1}(X_j)\big) + \big[\theta^1_i(\omega^{-1}(X_i)), \theta^1_j(\omega^{-1}(X_j))\big]_\mathfrak{g} \\
&= pr_{\mathfrak{g}_{i+j}} \circ d\omega\big(\omega^{-1}(X_i), \omega^{-1}(X_j)\big) + [X_i, X_j]_\mathfrak{g} \\
&= (\Omega^\omega)^0\big(\omega^{-1}(X_i), \omega^{-1}(X_j)\big).
\end{aligned}$$

We further know (see subsection 3.5.2) that the commutator respects the filtration of our bundle for $i, j < 0$, that is $[\Gamma(T^iE), \Gamma(T^jE)] \subset \Gamma(T^{i+j}E)$. Thus we have for the curvature and sections $\xi \in \Gamma(T^iE)$, $\eta \in \Gamma(T^jE)$ with $i, j < 0$:

$$\begin{aligned}
\Omega^\omega(\xi, \eta) &= d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]_\mathfrak{g} \\
&= \xi(\underbrace{\omega(\eta)}_{\in \mathfrak{g}^j \subset \mathfrak{g}^{i+j}}) - \eta(\underbrace{\omega(\xi)}_{\in \mathfrak{g}^i \subset \mathfrak{g}^{i+j}}) - \omega(\underbrace{[\xi, \eta]}_{\in \Gamma(T^{i+j}E)}) + \underbrace{[\omega(\xi), \omega(\eta)]_\mathfrak{g}}_{\in \mathfrak{g}^{i+j}} \\
&\in \mathfrak{g}^{i+j}.
\end{aligned}$$

Taking into consideration that the curvature is horizontal we finally obtain that $(\Omega^\omega)^j \equiv 0$ for $j < 0$. Thus the Cartan geometry is regular ($\partial^*\Omega^\omega \equiv 0$ and $(\Omega^\omega)^l \equiv 0$ for $l \le 0$) if and only if the underlying P-frame bundle of degree one satisfies the structure equations and is harmonic.

3.6 Tractor Bundles

In chapter 5 we will explain the construction of the Fefferman space according to [CG08]. It will be obtained using the Cartan bundle of the CR manifold. However using the tractor bundle of the CR manifold will result in a nice and helpful view of the Cartan bundle of the Fefferman space. So we now have to take care of tractor bundles, the correspondence between Cartan and tractor connections and the recovering of the Cartan bundle from the tractor bundle in the conformal case. This section is based on [CG02], [CS03] and [ScSl00].
Let $(\mathcal{G}, \pi, M; \omega)$ be a Cartan geometry of type (G, P), $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be a $|k|$-graded Lie algebra. With $\mathfrak{g}^j := \mathfrak{g}_j \oplus \mathfrak{g}_{j+1} \oplus \cdots \oplus \mathfrak{g}_k$ the filtration of the Lie algebra $\mathfrak{g} = \mathfrak{g}^{-k} \supset \mathfrak{g}^{-k+1} \supset \cdots \supset \mathfrak{g}^k = \mathfrak{g}_k$ gives a filtration of the tangent space $T\mathcal{G}$ by using the Cartan connection ω,

$$T^j\mathcal{G} := \omega^{-1}(\mathfrak{g}^j).$$

This filtration is right invariant under the action of the subgroup P according to the definition, $P = \{g \in G \mid Ad(g)\mathfrak{g}^j \subset \mathfrak{g}^j \text{ for all } j = -k, \dots, k\}$. Projection results in a filtration of the tangent space TM, $T^jM := d\pi(T^j\mathcal{G})$.

$$TM = T^{-k}M \supset T^{-k+1}M \supset \cdots \supset T^{-1}M \supset T^0M = \{0\}$$

The associated graded tangent space is defined by

$$Gr(TM) := \left(T^{-k}M/_{T^{-k+1}M}\right) \oplus \cdots \oplus \left(T^{-2}M/_{T^{-1}M}\right) \oplus T^{-1}M.$$

For regular Cartan geometries $(\mathcal{G}, \pi, M; \omega)$, i.e. the curvature of the Cartan connection $\partial^* \circ \Omega^\omega$ and $(\Omega^\omega)^l$ $(l \leq 0)$ vanish, we can define the graded commutator of vector fields $[\cdot,\cdot]^{Gr} : \Gamma\left(T^iM/_{T^{i+1}M}\right) \wedge \left(T^jM/_{T^{j+1}M}\right) \longrightarrow \Gamma\left(T^{i+j}M/_{T^{i+j+1}M}\right)$ in the following way. For vector fields $X \in \Gamma(T^iM)$ and $Y \in \Gamma(T^jM)$ we set

$$[X, Y]^{Gr} := [X, Y] + T^{i+j+1}M.$$

This is actually a section of $T^{i+j}M/_{T^{i+j+1}M}$ which can be seen by writing for the vector fields $X = d\pi \sum_r \lambda_r \omega^{-1}(X_r)$ with (X_r) being a basis of $\mathfrak{g}^i$ and in the same way with (Y_s) being a basis of $\mathfrak{g}^j$ we have $Y = d\pi \sum_s \mu_s \omega^{-1}(Y_s)$. Hence

$$[X, Y] = d\pi \left(\sum_{rs} \lambda_r \mu_s [\omega^{-1}(X_r), \omega^{-1}(Y_s)] \right) \ \ mod\, T^{i+j+1}M.$$

Due to the Cartan geometry being regular, we have especially for all $l < 0$ and all $a_{i/j} \in \mathfrak{g}_{i/j}$

$$0 = (\Omega^\omega)^l(\omega^{-1}(a_i), \omega^{-1}(a_j)) = -\omega_{i+j+l}\left(\left[\omega^{-1}(a_i), \omega^{-1}(a_j)\right]\right).$$

Thus $[X, Y]$ is a section in $T^{i+j}M$. And therefore $[X, Y]^{Gr} \in \Gamma\left(T^{i+j}M/_{T^{i+j+1}M}\right)$.
As we can see in the equation above the graded commutator of vector fields is $\mathcal{C}^\infty(M)$-linear, that is for all $f \in \mathcal{C}^\infty(M)$ we have $[fX, Y]^{Gr} = [X, fY]^{Gr} = f[X, Y]^{Gr}$. Furthermore note that for $Z \in T^{i+1}M$ it holds $[X + Z, Y]^{Gr} = [X, Y]^{Gr}$. So we obtain the graded commutator for vector fields of the associated graded tangent space

$$[\cdot,\cdot]^{Gr} : \Gamma\left(T^iM/_{T^{i+1}M}\right) \wedge \Gamma\left(T^jM/_{T^{j+1}M}\right) \longrightarrow \Gamma\left(T^{i+j}M/_{T^{i+j+1}M}\right).$$

Now any P-module $\mathbb{V}$ defines a bundle over the manifold M associated to the Cartan bundle, $\mathcal{V} := \mathcal{G} \times_P \mathbb{V}$. Let $\varrho : G \longrightarrow Gl(\mathbb{V})$ be a representation with $\varrho_* : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathbb{V})$ being injective, then we call the corresponding associated bundle $\mathcal{V} := \mathcal{G} \times_P \mathbb{V}$ a standard tractor bundle. We can view a standard tractor bundle as associated to the extended Cartan bundle $\widetilde{\mathcal{G}} = \mathcal{G} \times_P G$, i.e. $\mathcal{V} = \widetilde{\mathcal{G}} \times_{[G,\varrho]} \mathbb{V}$. Since the Cartan connection ω is uniquely extended to a G-principal bundle connection $\widetilde{\omega}$ on the bundle $\widetilde{\mathcal{G}}$

$$\widetilde{\omega}_{[u,g]} := Ad(g^{-1}) \circ (\pi_{\mathcal{G}}^* \omega) + \pi_G^* \omega_G$$

we obtain on $\mathcal{V} = \mathcal{G} \times_P \mathbb{V}$ the induced linear connection ∇^ω in the usual way:

$$\nabla^\omega_X [s, v] := \Big[s, X(v) + \varrho_*\big(\omega \circ ds(X)\big)v\Big]$$

with $X \in \mathfrak{X}(M)$, $s : U \subset M \longrightarrow \mathcal{G}$ a local section and $v \in \mathbb{V}$.

Thus every Cartan connection on $\mathcal{G}$ yields a linear connection on $\mathbb{V}$. The linear connections induced by Cartan connections are a special type of connections the so called tractor connections. To define tractor connections we have to consider the properties that distinguish the tractor connections from all linear connections on $\mathcal{V}$.

As we have seen in section 3.2 we have a grading element $E \in \mathfrak{g}_0$ with $ad(E)|_{\mathfrak{g}_j} = jId$ for every $j = -k, \ldots, k$. More precisely E is an element of the center of $\mathfrak{g}_0$ and therefore for every $g_0 \in G_0$ the Adjoint action of g_0 on E has to be trivial, $Ad(g_0)E = const = Ad(1)E = E$ for all $g_0 \in G_0$. Thus for an irreducible G_0-module E has to act by multiplication with a scalar. So we can split $\mathbb{V}$ according to the eigenvalues of the action of the grading element E, $\mathbb{V} = \oplus_\mu \mathbb{V}_\mu$. For $X \in \mathfrak{g}_j$ we have $X(\mathbb{V}_\mu) \subset \mathbb{V}_{\mu+j}$ since with $v \in \mathbb{V}_\mu$ we can write

$$\begin{aligned} E(X(v)) &= [E, X]_{\mathfrak{g}}(v) + X(E(v)) \\ &= jX(v) + X(\mu v) \\ &= (\mu + j)X(v). \end{aligned}$$

Especially the decomposition $\mathbb{V} = \oplus_\mu \mathbb{V}_\mu$ is G_0-invariant.

The filtration of $\mathbb{V}$ obtained via $\mathbb{V}^\mu := \oplus_{\nu \geq \mu} \mathbb{V}_\nu$ is P-invariant and for each eigenvalue μ of E acting on $\mathbb{V}$ we get a smooth subbundle $\mathcal{V}^\mu := \mathcal{G} \times_P \mathbb{V}^\mu \subset \mathcal{V}$. These subbundles form a decreasing filtration of $\mathcal{V}$.

For any element $u \in \mathcal{G}$ we define a map between $\mathbb{V}$ and the fibre $\mathcal{V}_x$ over the point $x = \pi(u)$ by setting

$$\begin{aligned} \underline{u} : \mathbb{V} &\longrightarrow \mathcal{V}_x \\ v &\mapsto [u, v] \in \mathcal{V} = \mathcal{G} \times_P \mathbb{V}. \end{aligned}$$

For any $p \in P$ it holds $\underline{R_p u}(v) = [R_p u, v] = [u, \varrho(p)v] = \underline{u}\big(\varrho(p)(v)\big)$. With the help of this map we can define an isomorphism between the smooth sections of the tractor bundle $\mathcal{V}$ and the P-equivariant maps from $\mathcal{G}$ to $\mathbb{V}$.

$$\begin{aligned} \Gamma(\mathcal{V}) &\longrightarrow C^\infty(\mathcal{G}, \mathbb{V})^{(\varrho,P)} = \{f : \mathcal{G} \longrightarrow \mathbb{V} \mid f \text{ is } P\text{-equivariant}\} \\ t &\mapsto \widetilde{t} \end{aligned}$$

$$\begin{aligned} \widetilde{t}(u) &:= \underline{u}^{-1} \circ t \circ \pi(u) \\ & \qquad \widetilde{t} \text{ is } P\text{-equivariant since} \\ \widetilde{t}(R_p u) &= (\underline{R_p u})^{-1} \circ t \circ \pi(R_p u) \\ &= \varrho(p^{-1}) \circ \underline{u}^{-1} \circ t \circ \pi(u) \\ &= \varrho(p^{-1}) \circ \widetilde{t}(u). \end{aligned}$$

And the other way round

$$\begin{array}{rcl} C^\infty(\mathcal{G},\mathbb{V})^{(\varrho,P)} & \longrightarrow & \Gamma(\mathcal{V}) \\ \widetilde{t} & \mapsto & t \\ & & t(x) \;:=\; \underline{u}\circ\widetilde{t}(u) \text{ for a } u\in\mathcal{G}_x \\ & & \quad\text{This is well defined since} \\ & & \quad \underline{R_p u}\circ\widetilde{t}(R_p u) \;=\; \underline{R_p u}\circ\varrho(p^{-1})\circ\widetilde{t}(u) \\ & & \quad \phantom{\underline{R_p u}\circ\widetilde{t}(R_p u)} \;=\; \underline{u}\circ\widetilde{t}(u) \\ & & \quad \phantom{\underline{R_p u}\circ\widetilde{t}(R_p u)} \;=\; t(x). \end{array}$$

Now let $\nabla : \Gamma(\mathcal{V}) \longrightarrow \Gamma(T^*M\otimes\mathcal{V})$ be a linear connection on $\mathcal{V}=\mathcal{G}\times_P\mathbb{V}$. With $u\in\mathcal{G}_x$, $\xi\in\mathfrak{X}(\mathcal{G})$ and $t\in\Gamma(\mathcal{V})$ we have $\big(\nabla_{d\pi\xi}t\big)_x\in\mathcal{V}_x$ and hence $\underline{u}^{-1}\big(\nabla_{d\pi\xi}t\big)_x\in\mathbb{V}$. Furthermore we have the P-equivariant map $\widetilde{t}:\mathcal{G}\longrightarrow\mathbb{V}$ giving $\xi(\widetilde{t})(u)\in\mathbb{V}$. For any smooth function $f\in C^\infty(M,\mathbb{R})$ we have

$$\begin{aligned} \underline{u}^{-1}\big(\nabla_{d\pi\xi}ft\big)_x-\xi(\widetilde{ft})(u) &= \underline{u}^{-1}\left(f(x)\big(\nabla_{d\pi\xi}t\big)_x+(d\pi\xi)(f)_x t(x)\right)-\xi\big((\pi^*f)\widetilde{t}\big)(u) \\ &= f(x)\underline{u}^{-1}\big(\nabla_{d\pi\xi}t\big)_x+(d\pi\xi)(f)_x\underbrace{\underline{u}^{-1}\circ t(x)}_{=\widetilde{t}(u)} \\ &\quad -(\xi)(\pi^*f)_u\widetilde{t}(u)-f(x)\xi\big(\widetilde{t}\big)(u) \\ &= f(x)\left(\underline{u}^{-1}\big(\nabla_{d\pi\xi}t\big)_x-\xi\big(\widetilde{t}\big)(u)\right). \end{aligned}$$

I.e. $\underline{u}^{-1}\big(\nabla_{d\pi\xi}t\big)_x-\xi\big(\widetilde{t}\big)(u)$ depends only on the value $t(x)$ or accordingly on the value $\widetilde{t}(u)$. Hence for every vector $\xi\in T_u\mathcal{G}$ we obtain a linear map $\Phi(\xi):\mathbb{V}\longrightarrow\mathbb{V}$ defined by $\Phi(\xi)\big(\widetilde{t}(u)\big)=\underline{u}^{-1}\big(\nabla_{d\pi\xi}t\big)_x-\xi\big(\widetilde{t}\big)(u)$ for any section $t\in\Gamma(\mathcal{V})$.

Definition 3.16 *Let ∇ be a linear connection on the tractor bundle $\mathcal{V}$. Then ∇ is called*

- *a $\mathfrak{g}$-connection if and only if for each tangent vector $\xi\in T_u\mathcal{G}$ the induced linear map $\Phi(\xi):\mathbb{V}\longrightarrow\mathbb{V}$ is given by the action of some element of $\mathfrak{g}$,*
- *nondegenerate if and only if for any point $x\in M$ and any nonzero tangent vector $\xi\in T_xM$ there exists a number μ and a (local) smooth section t of $\mathcal{V}^\mu$ such that $(\nabla_\xi t)_x\notin\mathcal{V}^\mu_x$,*
- *a tractor connection on $\mathcal{V}$ if it is a nondegenerate $\mathfrak{g}$-connection.*

Now we will see, that a Cartan connection on $\mathcal{G}$ induces a tractor connection on $\mathcal{V}$ and conversely as can be found in [CG02] and [CG03].

Proposition 3.7 *Let $\mathcal{G}\longrightarrow M$ be a Cartan bundle and $\mathcal{V}=\mathcal{G}\times_P\mathbb{V}$ a tractor bundle for $\mathcal{G}$.*

- *A Cartan connection ω on $\mathcal{G}$ induces a tractor connection on $\mathcal{V}$.*
- *Conversely, a tractor connection ∇ on $\mathcal{V}$ induces a Cartan connection ω on $\mathcal{G}$.*

Proof:

- Assume that we have given a Cartan connection $\omega\in\Omega^1(\mathcal{G},\mathfrak{g})$. The linear connection induced by ω is denoted by ∇ and we have for $\xi\in\mathfrak{X}(M)$ and $t\in\Gamma(\mathcal{V})$

$$\begin{array}{rcl} \nabla_\xi t : M & \longrightarrow & \mathcal{V}=\mathcal{G}\times_{[P,\varrho]}\mathbb{V} \\ x & \mapsto & \big[u,\overline{\xi}(\widetilde{t})_u+\varrho_*(\omega(\overline{\xi}))(\widetilde{t})_u\big], \end{array}$$

  where $\overline{\xi}\in\mathfrak{X}(\mathcal{G})$ is a lift of ξ and $\widetilde{t}\in C^\infty(\mathcal{G},\mathbb{V})^{(\varrho,P)}$ the P-equivariant map corresponding to t.

This linear connection is also a $\mathfrak{g}$-connection since for every tangent vector $\xi \in T_xM$ the term $\underline{u}^{-1}(\nabla_\xi \cdot) - \overline{\xi}(\widetilde{t})_u = \varrho(\omega(\overline{\xi}))(\widetilde{t})_u$ is obviously given by the action of $\omega(\overline{\xi}) \in \mathfrak{g}$.

It remains to show that ∇ is nondegenerate. Let again ξ be a vector in T_xM, $\xi \neq 0$. We choose a lift $\overline{\xi} \in T_u\mathcal{G}$ with $\omega(\overline{\xi}) \in \mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$, which can be obtained by adding a suitable fundamental vector field. Since ϱ is effective we have a $v \in \mathbb{V}$ such that $\varrho(\omega(\overline{\xi}))v \neq 0$. Without loss of generality we can assume v to be an eigenvector of the grading element E. Its eigenvalue may be denoted with μ. Now we choose a smooth section $t \in \Gamma(\mathcal{V}^\mu)$ with $\widetilde{t}(u) = \underline{u}^{-1}(t(x)) = v$. Then it holds $\overline{\xi}(\widetilde{t}))_u \in \mathbb{V}^\mu$ and consequently $\underline{u}^{-1}(\nabla_\xi t)_x \equiv \varrho\big(\omega(\overline{\xi})\big)(v)$ modulo $\mathbb{V}^\mu$. We can conclude

$$\varrho(E)\big(\varrho(\omega(\overline{\xi}))(v)\big) = \underbrace{\varrho(\omega(\overline{\xi}))}_{\in\mathfrak{g}_-} \circ \underbrace{\varrho(E)(v)}_{=\mu v} + \varrho\big(\underbrace{[E, \omega(\overline{\xi})]_{\mathfrak{g}}}_{\in\mathfrak{g}_-}\big)(v) \notin \mathbb{V}^\mu.$$

However this gives the result wanted

$$(\nabla_\xi t)_x = \big[u, \underbrace{\overline{\xi}(\widetilde{t})_u}_{\in\mathbb{V}^\mu} + \underbrace{\varrho(\omega(\overline{\xi}))(\widetilde{t})_u}_{\notin\mathbb{V}^\mu}\big] \notin \mathcal{V}^\mu_x.$$

Thus ∇ is nondegenerate and so a tractor connection.

- Assume now we have given a tractor connection ∇ on the tractor bundle $\mathcal{V}$. Let u be a point in $\mathcal{G}$ in the fibre over $x \in M$ and let $\xi \in T_u\mathcal{G}$ be an arbitrary vector. Since a tractor connection is especially a $\mathfrak{g}$-connection we have an element $\omega(\xi) \in \mathfrak{g}$ satisfying $\Phi(\xi) = \varrho_*\big(\omega(\xi)\big) : \mathbb{V} \longrightarrow \mathbb{V}$ which is more precisely

$$\varrho_*\big(\omega(\xi)\big)(\widetilde{t}(u)) = \underline{u}^{-1}\left(\nabla_{d\pi\xi} t\right)_x - \xi(\widetilde{t})(u) \text{ for all sections } t \in \Gamma(\mathcal{V}).$$

Thus we obtain a well defined map $\omega : T_u\mathcal{G} \longrightarrow \mathfrak{g}$ since the action $\varrho_* : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathbb{V})$ is injective.

At first we prove that ω is smooth. Of course the maps

$$\begin{aligned} \mathcal{G} \ni u &\mapsto d\pi(\xi)_u \in TM \\ \mathcal{G} \ni u &\mapsto \underline{u}^{-1}\left(\nabla_{d\pi\xi} t\right)_{\pi(u)} \in \mathbb{V} \\ \mathcal{G} \ni u &\mapsto \xi(\widetilde{t})_u \in \mathbb{V} \end{aligned}$$

are smooth. So the defining equation for ω gives smoothness of $\omega(\xi)$ for every smooth vector field ξ. Hence ω itself is smooth.

Now we prove that ω is actually a Cartan connection.

ω reproduces the generators of the fundamental vector fields

Let $\widetilde{X}$ be the fundamental vector field generated by $X \in \mathfrak{p}$. With $\widetilde{X}$ being horizontal the defining equation for ω simplifies and keeping in mind that $\widetilde{t}$ is P-equivariant we obtain:

$$\begin{aligned} \varrho_*\big(\omega(\widetilde{X})\big)(\widetilde{t}(u)) &= \underline{u}^{-1}\big(\nabla_{\underbrace{d\pi\widetilde{X}}_{=0}} t\big)_x - \widetilde{X}(\widetilde{t})_u \\ &= -\widetilde{X}(\widetilde{t})_u \\ &= -\ \tfrac{d}{dt}\big(\widetilde{t}(R_{\exp(tX)}u)\big)\big|_{t=0} \\ &= -\ \tfrac{d}{dt}\big(\varrho(\exp(-tX)) \circ \widetilde{t}(u)\big)\big|_{t=0} \\ &= \varrho_*(X)\widetilde{t}(u). \end{aligned}$$

With ϱ_* being injective this gives the result wanted, $\omega(\widetilde{X}) = X$ for all $X \in \mathfrak{p}$, that is to say ω reproduces the generators of the fundamental vector fields.

$\omega : T_u\mathcal{G} \longrightarrow \mathfrak{g}$ **is a linear isomorphism for every** $u \in \mathcal{G}$

Let $\xi \in T_u\mathcal{G}$ be a vector which is not vertical, $d\pi\xi \neq 0$. Since ∇ is nondegenerate we can find a number μ and a section $t \in \Gamma(\mathcal{V}^\mu)$ with $\big(\nabla_{d\pi\xi} t\big)_x \notin \mathcal{V}^\mu_x$. However this is equal to $\underline{u}\big(\xi(\widetilde{t})(u) + \varrho_*(\omega(\xi))(\widetilde{t}(u))\big)$ and $\widetilde{t}$ and also $\xi(\widetilde{t})$ have values in $\mathbb{V}^\mu$. Consequently $\omega(\xi)$ cannot vanish. In the case of ξ being vertical, $\omega(\xi)$ cannot vanish since ω reproduces the generators of the fundamental vector fields and is therefore injective on each vertical tangent space. Thus $\omega : T_u\mathcal{G} \longrightarrow \mathfrak{g}$ is injective and with $\mathcal{G}$ and $\mathfrak{g}$ being of the same dimension $\omega : T_u\mathcal{G} \longrightarrow \mathfrak{g}$ is an isomorphism.

ω **is** P**-equivariant**

Let u be a point in $\mathcal{G}$, $\xi \in T_u\mathcal{G}$ a vector and p an element of the Lie group P. Again we use the defining equation for ω.

$$\begin{aligned}\varrho_*\big((R_p^*\omega)_u(\xi)\big)\big(\widetilde{t}(R_p u)\big) &= \varrho_*\big(\omega_{R_p u}(dR_p\xi)\big)\big(\widetilde{t}(R_p u)\big)\\ &= \underline{R_p u}^{-1}\big(\nabla_{d\pi\circ dR_p\xi}t\big)_{\pi(R_p u)} - \big(dR_p\xi\big)(\widetilde{t})_{R_p u}\end{aligned}$$

Recall that we have $\underline{R_p u}^{-1} = \varrho(p^{-1}) \circ \underline{u}^{-1}$ and for the P-equivariant map $\widetilde{t}$ we have $(dR_p\xi)(\widetilde{t})_{R_p u} = \varrho(p^{-1}) \circ \xi(\widetilde{t})_u$. Thus we continue

$$\begin{aligned}\varrho_*\big((R_p^*\omega)_u(\xi)\big)\big(\widetilde{t}(R_p u)\big) &= \varrho(p^{-1}) \circ \big(\underline{u}^{-1}\big(\nabla_{d\pi\xi}t\big)_{\pi(u)} - \xi(\widetilde{t})_u\big)\\ &= \varrho(p^{-1}) \circ \big(\varrho_*(\omega_u(\xi))(\widetilde{t}(u))\big).\end{aligned}$$

ϱ is a left action on $\mathbb{V}$.

$$\begin{aligned}\varrho_*\big((R_p^*\omega)_u(\xi)\big)\big(\widetilde{t}(R_p u)\big) &= \varrho_*(dL_{p^{-1}} \circ \omega_u(\xi))(\widetilde{t}(u))\\ &= \varrho_*(Ad(p^{-1}) \circ \omega_u(\xi)) \circ \varrho(p^{-1})(\widetilde{t}(u))\\ &= \varrho_*(Ad(p^{-1}) \circ \omega_u(\xi))\big(\widetilde{t}(R_p u)\big)\end{aligned}$$

Again with ϱ_* being injective this is the result wanted, ω is P-equivariant, $(R_p^*\omega)_u = Ad(p^{-1}) \circ \omega_u$.

Summing up $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is a Cartan connection.

□

As known from principal bundles we have for the curvature of a linear connection ∇ induced by a principal bundle connection $\widetilde{\omega}$

$$\mathcal{R}^\nabla(\xi,\eta)[u,v] = \big[u, \varrho\big(\Omega^{\widetilde{\omega}}(\overline{\xi},\overline{\eta})_u\big)(v)\big].$$

With $\widetilde{\omega}$ being the principal bundle connection on $\widetilde{\mathcal{G}} = \mathcal{G} \times_P G$ induced by the Cartan connection ω we obtain for the curvature of the tractor connection ∇:

Proposition 3.8 *Let ∇ be a tractor connection on the tractor bundle $\mathcal{V} = \mathcal{G} \times_{[P,\varrho]} \mathbb{V}$ and ω the induced Cartan connection on the Cartan bundle $\mathcal{G}$. Then we have for the curvatures*

$$\mathcal{R}^\nabla(\xi,\eta)[u,v] = \big[u, \varrho\big(\Omega^{\omega}(\overline{\xi},\overline{\eta})_u\big)(v)\big],$$

where $\overline{\xi},\overline{\eta} \in \mathfrak{X}(\mathcal{G})$ are lifts of the vector fields $\xi,\eta \in \mathfrak{X}(M)$ and $[u,v] \in \mathcal{V}$.

3.6.1 Notations in the Conformal Case

We will now give a short overview on the notations we use for the Cartan bundle and connection in the conformal case. Generally we will use a tilde to identify objects of conformal manifolds. Details for this can be found for example in [Feh05].

We will restrict to the Riemannian case here in order to simplify the notations. However arbitrary signature poses no problem at all besides entraining the signs.

For a conformal manifold (M, c) of signature $(0, n)$ the bundle of all conformal reperes is denoted by $\widetilde{\mathcal{G}_0}$. This is a $\widetilde{G_0} = CO(n)$-principal bundle. The Lie group in the conformal case is $\widetilde{G}' = PO(1, n+1) = O(1, n+1)/_Z$, where Z is the center of the action of $O(1, n+1)$ on the line $\mathbb{R}f_-$, whose Lie algebra is endowed with a $|1|$-grading. This can be seen when using the basis $\left(f_- := \frac{1}{\sqrt{2}}(e_{p+q+1} - e_0), e_1, \ldots, e_{p+q}, \frac{1}{\sqrt{2}}(e_{p+q+1} + e_0)\right)$. The stabilizer of the null line $\ell := \mathbb{R}f_-$ is denoted by $\widetilde{P} = stab_\ell(PO(1, n+1))$ and is actually a subgroup of $O(1, n+1)$. Thus we can simply use the groups $\widetilde{G} = O(1, n+1)$ and $\widetilde{P} \subset \widetilde{G}$. Then we can identify the following subgroups and their Lie algebras.

$$
\begin{aligned}
\widetilde{G_0} = CO(n) \quad \text{with Lie algebra} \quad \widetilde{\mathfrak{g}_0} &= \left\{ \left(\begin{array}{ccc} -a & & \\ & A & \\ & & a \end{array} \right) \;\middle|\; \begin{array}{l} a \in \mathbb{R} \\ A \in \mathfrak{so}(n) \end{array} \right\} \\
\widetilde{P} = CO(n) \ltimes (\mathbb{R}^n)^* \quad \text{with Lie algebra} \quad \widetilde{\mathfrak{p}} &= \widetilde{\mathfrak{g}_0} \oplus \widetilde{\mathfrak{g}_1} \\
&= \widetilde{\mathfrak{g}_0} \oplus \left\{ \left(\begin{array}{ccc} 0 & x^t & 0 \\ & 0 & -x \\ & & 0 \end{array} \right) \;\middle|\; x^t \in (\mathbb{R}^n)^* \right\}.
\end{aligned}
$$

Let a be the conformal factor of an element $p \in \widetilde{P}$ and $v \in \ell$. Then we have for the action of p on v: $p \cdot v = a^{-1}v$. A conformal repere $u \in \widetilde{\mathcal{G}_0}$ is changed by elements of the conformal group by an orthogonal matrix and multiplication with the conformal factor. I.e. if g_x is a metric in T_xM such that the repere $u \in (\widetilde{\mathcal{G}_0})_x$ is orthogonal with respect to g_x, then the repere $R_{b_0}u$ will be orthogonal with respect to the metric $a^{-2}g_x$, where a is the conformal factor of b_0.

In the case of $n \geq 3$ the $\widetilde{G_0}$-principal bundle $\widetilde{\mathcal{G}_0} \longrightarrow M$ of all conformal reperes can be uniquely prolonged to a $\widetilde{P}$-principal bundle, $\widetilde{\mathcal{G}} \longrightarrow M$, endowed with a Cartan connection $\widetilde{\omega}$. The projections are denoted with $\widetilde{\mathcal{G}} \xrightarrow{\pi^1} \widetilde{\mathcal{G}_0} \xrightarrow{\pi_1} M$ and $\pi = \pi_1 \circ \pi^1$.

By fixing a metric $g \in c$ we obtain a global section

$$
\begin{aligned}
\sigma_g : \widetilde{\mathcal{G}_0} &\longrightarrow \widetilde{\mathcal{G}} \\
u &\mapsto ker(A^g_u) \text{ where } A^g \text{ is the Levi Civita connection}
\end{aligned}
$$

and in this realization of the Cartan bundle the Cartan connection $\widetilde{\omega}$ is characterized by $\widetilde{\omega}_{-1} := pr_{\widetilde{\mathfrak{g}_{-1}}} \circ \widetilde{\omega}$ is the pull back of the displacement form of the conformal repere bundle, $\sigma_g^*\widetilde{\omega}_0$ is the Levi Civita connection with respect to the metric g and $\sigma_g^*\widetilde{\omega}_1$ corresponds to the Schouten tensor. This Cartan connection is the normal Cartan connection, which is uniquely defined by the conformal structure up to isomorphism. In the next subsection we will use this realization of the conformal Cartan bundle and the normal Cartan connection to identify the normal tractor connection.

3.6.2 Tractor Bundles for Conformal Manifolds

This subsection is based on [CG03]. For a conformal manifold (M, c) of signature (p, q) we have a smooth subbundle $\mathcal{Q} \subset S^2T^*M$ with fibres isomorphic to $\mathbb{R}_+$ given by the values of all metrics $g \in c$. The sections of the ray bundle $\mathcal{Q}$ are the metrics in the conformal class c. With the projection $\pi_{\mathcal{Q}} : \mathcal{Q} \longrightarrow M$ this is a $\mathbb{R}_+$-principal bundle endowed with the principal action $\varrho(s)(g_x) := s^2 g_x$ for $s \in \mathbb{R}_+$ and $g_x \in \mathcal{Q}_x$ being an element in the fibre over $x \in M$. This action is defined following the convention of rescaling metrics by multiplication with the square of a function, $\hat{g} := f^2 g$.
For $w \in \mathbb{R}$ we set $\mathcal{E}[w]$ to be associated to the ray bundle $\mathcal{Q}$ in the following way

$$\begin{aligned} \mathcal{E}[w] \ :=& \ \mathcal{Q} \times_{[\mathbb{R}_+, s^{-w}]} \mathbb{R} \\ & [g_x, \lambda] = [s^2 g_x, s^w \lambda]. \end{aligned}$$

The sections of $\mathcal{E}[w]$ can be identified with the $\mathbb{R}_+$-equivariant maps on $\mathcal{Q}$:

$$\begin{array}{rcl} \Gamma(\mathcal{E}[w]) & \longrightarrow & \big\{ f \in C^\infty(\mathcal{Q}, \mathbb{R}) \ \big| \ f(s^2 g_x) = s^w f(g_x) \big\} \\ \big(x \mapsto [g_x, \lambda]\big) & \mapsto & \big(f : g_x \mapsto \lambda\big). \end{array}$$

Remark 3.5 *The associated vector bundle $\mathcal{E}[-n]$ can be identified with the basic density bundle containing the volume forms $vol(g)$ for $g \in c$ which transform in the following way: $vol(f^2 g) = f^n vol(g)$. Therefore, the bundles $\mathcal{E}[w]$ are also called density bundles.*

Now let $(\widetilde{\mathcal{G}}, \pi_{\widetilde{\mathcal{G}}}, M; \widetilde{\omega})$ be the Cartan bundle of (M, c) with the normal Cartan connection $\widetilde{\omega}$. Then the standard tractor bundle is defined as $\mathcal{T} := \widetilde{\mathcal{G}} \times_{\widetilde{P}} \mathbb{R}^{p+1,q+1}$ and the null line ℓ defines the subbundle $\mathcal{T}^1 := \widetilde{\mathcal{G}} \times_{\widetilde{P}} \ell$.
The subbundle $\mathcal{T}^1$ is isomorphic to the density bundle $\mathcal{E}[-1]$. To see this we fix a null vector $v_0 \in \ell$ and denote with g_x^u the metric for which $\pi_1(u) \in \mathcal{G}_0$ is an orthogonal repere. Then the isomorphism is given by

$$\begin{array}{ccc} \mathcal{T}^1 = \mathcal{G} \times_P \ell & \longrightarrow & \mathcal{E}[-1] = \mathcal{Q} \times_{[\mathbb{R}_+, s^{-1}]} \mathbb{R} \\ [u_x, \lambda v_0] & \mapsto & [g_x^u, \lambda] \\ \| & & \| \\ [R_p u, \underbrace{p^{-1} \lambda v_0}_{= a\lambda v_0}] & & [\underbrace{a^{-2} g_x^u}_{= g_x^{R_p u}}, a\lambda], \end{array}$$

where a is the conformal factor of $p \in \widetilde{P}$ as explained in the subsection above.

Later on we have to identify the conformal structure of a given standard tractor bundle. So we will now discuss how to recover from a given standard tractor bundle over a manifold M the conformal structure c on M and the Cartan bundle. Thus let us now consider a rank $p + q + 2$ real vector bundle $\pi : \mathcal{T} \longrightarrow M$ over the smooth manifold M of dimension $p + q \geq 3$. Further let $\mathcal{T}$ be endowed with a bundle metric h of signature $(p + 1, q + 1)$. Assume that we have also given an injective bundle map $i : \mathcal{E}[-1] \longrightarrow \mathcal{T}$ and denote its image, which is supposed to be null, with $\mathcal{T}^1 := i(\mathcal{E}[-1])$. Please note, that by fixing a metric g in the conformal class (once that we have identified it) the smooth sections of $\mathcal{T}^1$ can be interpreted as smooth maps on M with values in $\mathbb{R}$.

$$\begin{array}{rcl} \Gamma(\mathcal{T}^1) \simeq \Gamma(\mathcal{E}([-1]) \simeq \Gamma\big(\mathcal{Q} \times_{[\mathbb{R}_+, s^{-1}]} \mathbb{R}\big) & \overset{g \text{ fix}}{\longrightarrow} & C^\infty(M) \\ [g_x, a_x] & \mapsto & f(x) := a_x \end{array}$$

Then we denote with $\mathcal{T}^0$ the orthogonal complement of $\mathcal{T}^1$ in $\mathcal{T}$ with respect to the bundle metric h and obtain a filtration $\mathcal{T}^{-1} := \mathcal{T} \supset \mathcal{T}^0 \supset \mathcal{T}^1$. For conformal geometries the grading element is $E = diag(1, 0, \ldots, 0, -1)$ (see for example [Feh05]). So eigenvectors for the action of E on $\mathbb{R}^{p+1,q+1}$ are

- f_- with eigenvalue 1,
- $e_1, \ldots, e_n$ with eigenvalue 0 and
- f_+ with eigenvalue -1.

Hence the filtration of $\mathcal{T}$ according to the eigenvalues of E is exactly the same as obtained by forming the orthogonal complement of $\mathcal{T}^1$.

Lemma 3.13 *Having given a rank $p+q+2$ vector bundle $\mathcal{T} \longrightarrow M$ with a bundle metric h and a light like subbundle $\mathcal{T}_1 \subset \mathcal{T}$, which is given as the image of an injective bundle map $i : \mathcal{E}[-1] \longrightarrow \mathcal{T}$, then a linear connection ∇ is a tractor connection if and only if it is metric with respect to the bundle metric h and for every point $x \in M$ and all nonzero vector fields $\xi \in \mathfrak{X}(M)$ we have a section $\sigma \in \Gamma(\mathcal{T}^1)$ with $(\nabla_\xi \sigma)_x \notin \mathcal{T}^1_x$.*

Proof: A linear connection ∇ on $\mathcal{T}$ is a $\mathfrak{o}(p+1, q+1)$-connection if and only if it preserves the bundle metric, $\nabla h \equiv 0$. To see this we use the equivalence of the sections X of $\mathcal{T}$ and the P-equivariant maps $\widetilde{X} := \underline{u}^{-1} \circ X \circ \pi : \mathcal{G} \longrightarrow \mathbb{V}$. The linear connection ∇ being actually a $\mathfrak{o}(p+1, q+1)$-connection means that $\Phi(\xi)$, defined by $\Phi(\xi)\widetilde{X} = \widetilde{\nabla_\xi X} - \xi(\widetilde{X})$, is given by the action of some element in $\mathfrak{o}(p+1, q+1)$ for $\xi \in \mathfrak{X}(M)$. Thus we have for any sections $X, Y \in \Gamma(\mathcal{T})$

$$\langle \Phi(\xi)\widetilde{X}, \widetilde{Y} \rangle_{p+1,q+1} + \langle \widetilde{X}, \Phi(\xi)\widetilde{Y} \rangle_{p+1,q+1} \equiv 0.$$

However this is equivalent to

$$\begin{aligned}
\xi\big(h(X,Y)\big) &= \xi\big(\langle \widetilde{X}, \widetilde{Y} \rangle_{p+1,q+1}\big) \\
&= \langle \xi(\widetilde{X}), \widetilde{Y} \rangle_{p+1,q+1} + \langle \widetilde{X}, \xi(\widetilde{Y}) \rangle_{p+1,q+1} \\
&= \langle \xi(\widetilde{X}) + \Phi(\xi)(\widetilde{X}), \widetilde{Y} \rangle_{p+1,q+1} + \langle \widetilde{X}, \xi(\widetilde{Y}) + \Phi(\xi)(\widetilde{Y}) \rangle_{p+1,q+1} \\
&= \langle \widetilde{\nabla_\xi X}, \widetilde{Y} \rangle_{p+1,q+1} + \langle \widetilde{X}, \widetilde{\nabla_\xi Y} \rangle_{p+1,q+1} \\
&= h(\nabla_\xi X, Y) + h(X, \nabla_\xi Y).
\end{aligned}$$

Thus the linear connection ∇ is a $\mathfrak{o}(p+1, q+1)$-connection if and only if it preserves the bundle metric h.
Now a linear connection ∇ being nondegenerate means according to the definition that we have for all points $x \in M$ and all nonzero vector fields $\xi \in \mathfrak{X}(M)$ a section $\sigma \in \Gamma(\mathcal{T}^1)$ with $(\nabla_\xi \sigma)_x \notin \mathcal{T}^1_x$ or a section $\sigma \in \Gamma(\mathcal{T}^0)$ with $(\nabla_\xi \sigma)_x \notin \mathcal{T}^0_x$. We will prove that we can always find a section of the second type if a linear connection is nondegenerate. Let us assume ∇ preserves the bundle metric h and we have a point $x \in M$ and a vector field $\xi \in \mathfrak{X}(M)$ such that for every section $\sigma \in \Gamma(\mathcal{T}^1)$ we have $(\nabla_\xi \sigma)_x \in \mathcal{T}^1_x$. So for any sections $\sigma \in \Gamma(\mathcal{T}^1)$ and $\gamma \in \Gamma(\mathcal{T}^0)$ we have $h(\sigma, \gamma) \equiv 0$ and therefore

$$0 = \xi\big(h(\sigma, \gamma)\big) = h(\nabla_\xi \sigma, \gamma) + h(\sigma, \nabla_\xi \gamma).$$

However the first summand vanishes since $\nabla_\xi \sigma \in \mathcal{T}^1$ and $\gamma \in \mathcal{T}^0 = (\mathcal{T}^1)^\perp$. Thus $\nabla_\xi \gamma$ has to be orthogonal to σ, that means it is an element of $\mathcal{T}^0$, and ∇ cannot be nondegenerate in this case. We conclude, that a linear connection preserving the bundle metric h is nondegenerate if and only if for every point $x \in M$ and all nonzero vector fields $\xi \in \mathfrak{X}(M)$ we have a section $\sigma \in \Gamma(\mathcal{T}^1)$ with $(\nabla_\xi \sigma)_x \notin \mathcal{T}^1_x$.

□

We now want to take a closer look at $\mathcal{T}^0/_{\mathcal{T}^1}$. First of all let $\sigma_0 : U \longrightarrow \mathcal{T}^1$ be a local nonvanishing section of $\mathcal{T}^1$. Then $h(\sigma_0, \sigma_0)$ vanishes since $\mathcal{T}^1$ is a null subbundle and therefore also $h(\nabla_\xi \sigma_0, \sigma_0) = \frac{1}{2}\xi\big(h(\sigma_0, \sigma_0)\big)$ is zero for any vector field $\xi \in \mathfrak{X}(M)$. Consequently $\nabla_\xi \sigma_0 \in \Gamma(\mathcal{T}^0)$. This is also true for any section $\sigma \in \Gamma(\mathcal{T}^1)$ since we can write $\sigma = f\sigma_0$ for a real map f and

$$\begin{aligned} 0 &= f \cdot h(\nabla_\xi \sigma_0, \sigma_0) + \xi(f) \cdot h(\sigma_0, \sigma_0) \\ &= h\big(\nabla_\xi(f\sigma_0), \sigma_0\big) \\ &= h\big(\nabla_\xi(\sigma), \sigma_0\big). \end{aligned}$$

Thus we have $\nabla_\xi \sigma \in \Gamma(\mathcal{T}^0)$ for all sections $\sigma \in \Gamma(\mathcal{T}^1)$ and all vector fields $\xi \in \mathfrak{X}(M)$ and we can consider the following map:

$$\begin{aligned} \phi : \Gamma(TM \otimes \mathcal{T}^1) &\longrightarrow \Gamma\big(\mathcal{T}^0/_{\mathcal{T}^1}\big) \\ \xi \otimes \sigma &\mapsto [\nabla_\xi \sigma]. \end{aligned}$$

With $\phi(\xi \otimes f\sigma) = [\nabla_\xi f\sigma] = [f\nabla_\xi \sigma + \underbrace{\xi(f)\sigma}_{\in\Gamma(\mathcal{T}^1)}] = [f\nabla_\xi \sigma]$ we see that ϕ is bilinear over smooth functions. Thus ϕ is induced by a bundle map $\phi : TM \otimes \underbrace{\mathcal{E}[-1]}_{\simeq\mathcal{T}^1} \longrightarrow \mathcal{T}^0/_{\mathcal{T}^1}$. This map is injective on each fibre since the tractor connection ∇ is nondegenerate. Furthermore the bundles are of the same rank so ϕ is actually a bundle isomorphism,

$$TM \otimes \mathcal{E}[-1] \overset{\phi}{\simeq} \mathcal{T}^0/_{\mathcal{T}^1}.$$

Due to h being degenerate on $\mathcal{T}^0 = (\mathcal{T}^1)^\perp \subset \mathcal{T}$ with null space $\mathcal{T}^1$ we obtain a bundle metric of signature (p, q) on $\mathcal{T}^0/_{\mathcal{T}^1}$ induced by h. So we can define for vectors $X, Y \in T_xM$ and $\varphi \in \mathcal{E}[-1]_x$

$$h_\varphi(X, Y) := h\big(\phi(X \otimes \varphi), \phi(Y \otimes \varphi)\big).$$

This defines a conformal structure $h := [h_\varphi]$ on M.

In order to see that this is actually the conformal structure which defines the tractor bundle above we will first of all show that the Cartan bundle constructed by prolonging the conformal repere bundle as explained in the subsection above is isomorphic to

$$\mathcal{G} := \big\{\gamma_x : \mathbb{R}^{p+1,q+1} \longrightarrow \mathcal{T}_x \bigm| x \in M, \gamma \text{ orthogonal}, \gamma(\ell) \subset \mathcal{T}^1_x\big\},$$

where $\mathbb{R}^{p+1,q+1}$ is equipped with the inner product $\langle\cdot,\cdot\rangle_{p+1,q+1}$ and $\mathcal{T}$ with the bundle metric h. However this bundle is an adapted frame bundle for the bundle $\mathcal{T}$ as we will discuss next according to [CG00].

The conformal repere bundle can be written as

$$\widetilde{\mathcal{G}_0} = \big\{u_x : \widetilde{\mathfrak{g}_{-1}} \longrightarrow T_xM \bigm| x \in M, u_x \text{ conformal isomorphism}\big\},$$

with $\widetilde{\mathfrak{g}_{-1}} \simeq \mathbb{R}^{p,q}$ being endowed with the inner product $\langle\cdot,\cdot\rangle_{p,q}$ and M with the conformal structure $[h_\varphi]$. The Cartan bundle of $(M, [h_\varphi])$ is defined as the first prolongation of the conformal repere bundle, that is to say the bundle consisting of the torsion free, horizontal subspaces of the tangent space $T_u\widetilde{\mathcal{G}_0}$:

$$\widetilde{\mathcal{G}} := \big\{H \subset T_u\widetilde{\mathcal{G}_0} \bigm| u \in \widetilde{\mathcal{G}_0}, H \text{ horizontal}, t(H) = 0\big\}.$$

We have the canonical projection $\pi^1 : \widetilde{\mathcal{G}} \longrightarrow \widetilde{\mathcal{G}_0}$ and by fixing a metric h_φ we obtain a section $\widetilde{\sigma} : \widetilde{\mathcal{G}_0} \longrightarrow \widetilde{\mathcal{G}}$ by choosing for every conformal repere the horizontal space defined by the Levi Civita connection of h_φ.

Let us now consider the bundle

$$\mathcal{G} := \left\{ \gamma_x : (\mathbb{R}^{p+1,q+1}, \langle\cdot,\cdot\rangle_{p+1,q+1}) \longrightarrow (\mathcal{T}_x, h) \;\middle|\; x \in M, \gamma \text{ orthogonal}, \gamma(\ell) \subset \mathcal{T}_x^1 \right\}.$$

The P-action on $\mathcal{G}$ is defined by $R_p\gamma := \gamma \circ p$ using the action of P on $\mathbb{R}^{p+1,q+1}$. Please note that $\mathcal{G}$ is a subbundle of the frame bundle of the tractor bundle $\mathcal{T}$ and inherits its smooth structure in this way to form a P-principal bundle. If we fix a frame $(e_1, \dots, e_n)$ of $\widetilde{\mathfrak{g}_{-1}} \simeq \mathbb{R}^{p,q}$ and extend it by the null vectors $f_-, f_+ \in \mathbb{R}^{p+1,q+1}$ to form a frame of $\mathbb{R}^{p+1,q+1}$, we can interpret $\widetilde{\mathfrak{g}_{-1}} \simeq \ell^\perp/_\ell = span\{f_-, e_1, \dots, e_n\}/_{\mathbb{R}f_-}$. Thus restricting an orthogonal map $\gamma : (\mathbb{R}^{p+1,q+1}, \langle\cdot,\cdot\rangle_{p+1,q+1}) \longrightarrow (\mathcal{T}_x, h)$ with $\gamma(\ell) \subset \mathcal{T}_x^1$ to $\ell^\perp/_\ell$ gives a map $\gamma|_{\ell^\perp/_\ell} : \widetilde{\mathfrak{g}_{-1}} \longrightarrow \mathcal{T}_x^0/_{\mathcal{T}_x^1}$. And for each γ we obtain via

$$\begin{aligned} \widetilde{\gamma} : \widetilde{\mathfrak{g}_{-1}} \simeq \ell^\perp/_\ell &\longrightarrow T_xM \\ v &\mapsto \alpha \cdot X \\ &\quad\;\; \text{with } \phi^{-1} \circ \gamma(v) = X \otimes \alpha \in T_xM \otimes \mathcal{E}[-1] \end{aligned}$$

a conformal map $\widetilde{\gamma} : (\widetilde{\mathfrak{g}_{-1}}, \langle\cdot,\cdot\rangle_{p,q}) \longrightarrow (T_xM, [h_\varphi])$. Recall that by fixing a metric $h_\varphi \in [h_\varphi]$ we can identify the sections of $\mathcal{T}^1$ with $C^\infty(M)$. So finally we can define the projected curve $\widetilde{\pi^1}(\gamma) := \gamma(f_-) \cdot \widetilde{\gamma} \in \widetilde{\mathcal{G}_0}$, interpreting $\gamma(f_-) \in \mathbb{R}$, which gives the projection $\widetilde{\pi^1} : \mathcal{G} \longrightarrow \widetilde{\mathcal{G}_0}$. The other way round a section $\sigma : \widetilde{\mathcal{G}_0} \longrightarrow \mathcal{G}$ is gained using the fixed metric h_φ. A conformal map $u : (\mathfrak{g}_{-1}, \langle\cdot,\cdot\rangle_{p,q}) \longrightarrow (T_xM, [h_\varphi])$ can be split into an orthogonal map A with respect to h_φ and a conformal factor a. Thus we set $\gamma|_{\ell^\perp/_\ell} := u$, $\gamma(f_-) := a$, again using the identification $\mathcal{T}_x^1 \simeq_{h_\varphi} \mathbb{R}$, and complete this to be an orthogonal map $\gamma : \mathbb{R}^{p+1,q+1} \longrightarrow \mathcal{T}_x$.
Thus we obtain the following picture and the isomorphism $\Phi^{h_\varphi} : \mathcal{G} \longrightarrow \widetilde{\mathcal{G}}$, which is defined by $\Phi^{h_\varphi}(\gamma) := R_g \circ \sigma_{conf} \circ \widetilde{\pi^1}(\gamma)$ where $g \in P$ is the unique group element with $\gamma = R_g \circ \sigma \circ \widetilde{\pi^1}(\gamma)$.

$$\begin{tikzcd}
\mathcal{G} \arrow[rr, "\Phi^{h_\varphi}"] \arrow[dr, "\widetilde{\pi}^1"] & & \widetilde{\mathcal{G}} \arrow[dl, "\pi^1"'] \\
& \widetilde{\mathcal{G}_0} \arrow[ul, bend left, "\sigma"] \arrow[ur, bend right, "\widetilde{\sigma}"'] &
\end{tikzcd}$$

Φ^{h_φ} is an injective map between P-principal bundles over M and thus an isomorphism of the P-principal bundles. Hence $\mathcal{G}$ is the canonical Cartan bundle of the conformal manifold $(M, [h_\varphi])$ which is unique up to isomorphisms. Since the Cartan bundle $\mathcal{G}$ is constructed in a way that $\mathcal{T}$ is a standard tractor bundle of $\mathcal{G}$, we obtain that $\mathcal{T}$ is also a standard tractor bundle of $\widetilde{\mathcal{G}}$. If the given tractor connection ∇ corresponds to the normal Cartan connection $\widetilde{\omega}$, then $(\mathcal{T}, \nabla, h)$ will be the normal standard tractor bundle of the conformal manifold $(M, [h\varphi])$.
The normal Cartan connection $\widetilde{\omega}$ has to be torsion free and the Ricci-type trace of the $\widetilde{\mathfrak{g}}_0$-component of the curvature has to vanish, $\sum_i B_{\widetilde{\mathfrak{g}}}\left(\left[\Omega_0^{\widetilde{\omega}}\left(\widetilde{\omega}^{-1}(X_i), \widetilde{\omega}^{-1}(X)\right), Z^i\right]_{\widetilde{\mathfrak{g}}}, Y \right) \overset{!}{\equiv} 0$ for all $X, Y \in \widetilde{\mathfrak{g}_{-1}}$, where $X_1, \dots, X_n$ is a basis of $\widetilde{\mathfrak{g}_{-1}}$ and $Z^1, \dots, Z_n$ is the basis of $\widetilde{\mathfrak{g}_1}$ which is dual with respect to the Killing form $B_{\widetilde{\mathfrak{g}}}$. I.e. $\sum_i \left[\Omega_0^{\widetilde{\omega}}\left(\widetilde{\omega}^{-1}(X_i), \widetilde{\omega}^{-1}(\cdot)\right), Z^i\right]_{\mathfrak{g}}$ has to vanish. Now for conformal manifolds the torsion of a Cartan connection is the $\widetilde{\mathfrak{g}_{-1}}$-component of its curvature (see for example [Feh05]). Proposition 3.8 describes the relation between the curvatures of a Cartan connection $\widetilde{\omega}$ and its corresponding tractor connection ∇, for tractors $[u, v] \in \mathcal{T}_x$ and vectors $\xi, \eta \in T_xM$ we have $\mathcal{R}^\nabla(\xi, \eta)[u, v] = \left[u, \varrho\left(\Omega^{\widetilde{\omega}}(\overline{\xi}, \overline{\eta})_u\right)(v)\right]$

with $\overline{\xi}, \overline{\eta} \in T_u\widetilde{\mathcal{G}}$ being lifts of ξ respectively η. Since the action of the subalgebra $\widetilde{\mathfrak{p}} = \widetilde{\mathfrak{g}_0} \oplus \widetilde{\mathfrak{g}_1}$ leaves the null line $\ell \subset \mathbb{R}^{p+1,q+1}$ invariant, requesting the torsion of the Cartan connection to vanish is equivalent to demanding the action of the curvature of the tractor connection on the tractor bundle to preserve the subbundle $\mathcal{T}^1 \subset \mathcal{T}$.
We summarize the conclusions above.

Proposition 3.9 *Let M be a smooth manifold of dimension $n \geq 3$ and $(\mathcal{T}, \pi, M; h, \nabla)$ a tractor bundle with tractor connection ∇.*
Then we obtain $(\mathcal{T}^1)^{\perp}/_{\mathcal{T}^1} \simeq TM \otimes \mathcal{E}[-1]$ and $(\mathcal{T}, h, \nabla)$ is a standard tractor bundle for the manifold M endowed with the conformal structure defined by the restriction of the bundle metric h to $\big((\mathcal{T}^1)^{\perp}/_{\mathcal{T}^1}\big) \times \big((\mathcal{T}^1)^{\perp}/_{\mathcal{T}^1}\big)$. The tractor connection ∇ is normal if and only if the action of its curvature on the tractor bundle preserves the subbundle $\mathcal{T}^1$ and the Ricci-type trace of the $\mathfrak{g}_0$-component of the curvature, $\sum_i \left[\Omega_0^{\omega}\big(\omega^{-1}(X_i), \omega^{-1}(\cdot)\big), Z^i\right]_{\mathfrak{g}}$, vanishes.

Chapter 4

The Cartan Geometry of CR Manifolds

For constructing the Cartan boundary of a CR manifold we first need to set up the Cartan geometry of CR manifolds. Hence we will now apply the theory discussed in chapter 3 to CR manifolds, starting of with the specific Lie group and Lie algebra.

4.1 The Lie group $SU(p+1, q+1)$

Let $n = p+q$. With $\mathbb{C}^{p,q}$ we denote the complex vector space $\mathbb{C}^n$ joined by the hermitian product $\langle\cdot,\cdot\rangle_{p,q}$ of signature (p,q). Let $x = (x_1, \dots, x_{p+q})^t$ and $y = (y_1, \dots, y_{p+q})^t$ be two vectors of $\mathbb{C}^{p+q}$. Then we have

$$\langle x, y\rangle_{p,q} = x^* \cdot J^{p,q} \cdot y = x^* \cdot \begin{pmatrix} -I_p & \\ & I_q \end{pmatrix} \cdot y = -\sum_{i=1}^{p} \overline{x_i} y_i + \sum_{i=p+1}^{q+p} \overline{x_i} y_i.$$

The standard basis $(e_1, \dots, e_n)$ is therefore a unitary basis with respect to the hermitian product $\langle\cdot,\cdot\rangle_{p,q}$ and we set $\varepsilon_i := \langle e_i, e_i\rangle_{p,q} = \pm 1$.

We denote by $SU(p+1, q+1)$ the group of all special unitary transformations of $\mathbb{C}^{p+1,q+1}$, i.e. the group of all transformations of $\mathbb{C}^{p+1,q+1}$, which preserve the hermitian product $\langle\cdot,\cdot\rangle_{p+1,q+1}$ and whose determinant is one.

$$SU(p+1,q+1) := \left\{ A \in Gl(p+q+2, \mathbb{C}) \;\middle|\; \begin{array}{l} det(A) = 1 \text{ and} \\ \langle Ax, Ay\rangle_{p+1,q+1} = \langle x, y\rangle_{p+1,q+1} \\ \text{for all } x, y \in \mathbb{C}^{p+1,q+1} \end{array} \right\}$$

Let $(e_0, \underbrace{e_1, \dots, e_{p+q}}_{\in \mathbb{C}^{p,q}}, e_{p+q+1})$ be a unitary basis of $\mathbb{C}^{p+1,q+1} = \mathbb{C} \oplus \mathbb{C}^{p,q} \oplus \mathbb{C}$. We define two light like vectors $f_- := \frac{1}{\sqrt{2}}(e_{p+q+1} - e_0)$ and $f_+ := \frac{1}{\sqrt{2}}(e_{p+q+1} + e_0)$. With respect to the basis $(f_-, e_1, \dots, e_{p+q}, f_+)$ the hermitian product $\langle\cdot,\cdot\rangle_{p+1,q+1}$ has the matrix:

$$\langle\cdot,\cdot\rangle_{p+1,q+1} \hat{=} S := \begin{pmatrix} 0 & 0 & 1 \\ 0 & J^{p,q} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \text{ where } J^{p,q} := \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}.$$

Using this basis and the notation $A^* := \overline{A}^t$, the transposed and complex conjugated matrix, the group $SU(p+1, q+1)$ can be written as:

$$SU(p+1,q+1) = \{A \in M(p+q+2, \mathbb{C}) \mid det(A) = 1 \text{ and } A^* \cdot S \cdot A = S\}.$$

One of the important subgroups of $SU(p+1, q+1)$ needed here is the stabilizer P of the complex line $\mathbb{C}f_-$, that is $P := Stab(\mathbb{C}f_-) = \{A \in SU(p+1,q+1) \mid Af_- \in \mathbb{C}f_-\}$. Hence if we write

$$A = \begin{pmatrix} a & x^t & b \\ y & B & z \\ c & w^t & d \end{pmatrix} \quad \begin{array}{l} \text{with } a, b, c, d \in \mathbb{C},\ w, x, y, z \in \mathbb{C}^n,\ B \in M(n, \mathbb{C}) \\ \text{we obtain } y = 0, c = 0, A^* \cdot S \cdot A = S \text{ and } det(A) = 1. \end{array}$$

$$
\begin{aligned}
A^* \cdot S \cdot A &= \begin{pmatrix} \overline{a} & 0 & 0 \\ \overline{x} & B^* & \overline{w} \\ \overline{b} & \overline{z}^t & \overline{d} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & J^{p,q} & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & x^t & b \\ 0 & B & z \\ 0 & w^t & d \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & \overline{a} \\ \overline{w} & B^* J^{p,q} & \overline{x} \\ \overline{d} & \overline{z}^t J^{p,q} & \overline{b} \end{pmatrix} \cdot \begin{pmatrix} a & x^t & b \\ 0 & B & z \\ 0 & w^t & d \end{pmatrix} \\
&= \begin{pmatrix} 0 & \overline{a} w^t & \overline{a} d \\ a\overline{w} & \overline{w} \cdot x^t + \overline{x} \cdot w^t + B^* J^{p,q} B & b\overline{w} + B^* J^{p,q} z + d\overline{x} \\ a\overline{d} & \overline{d} x^t + \overline{z}^t J^{p,q} B + \overline{b} w^t & \overline{d} b + \overline{b} d + \langle z, z \rangle_{p,q} \end{pmatrix} \\
&\overset{!}{=} \begin{pmatrix} 0 & 0 & 1 \\ 0 & J^{p,q} & 0 \\ 1 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

So we obtain especially $d = \overline{a}^{-1}$ and therefore $a \neq 0$. Consequently $w = 0$ and the equation can be simplified as follows

$$
\begin{aligned}
A^* \cdot S \cdot A &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & B^* J^{p,q} B & B^* J^{p,q} z + \overline{a}^{-1} \overline{x} \\ 1 & a^{-1} x^t + \overline{z}^t J^{p,q} B & a^{-1} b + \overline{b} \overline{a}^{-1} + \langle z, z \rangle_{p,q} \end{pmatrix} \\
&\overset{!}{=} \begin{pmatrix} 0 & 0 & 1 \\ 0 & J^{p,q} & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

With $det(A) = a \cdot det \begin{pmatrix} B & z \\ 0 & \overline{a}^{-1} \end{pmatrix} = a \cdot \overline{a}^{-1} \cdot det(B)$ we get

$$
P = \left\{ \begin{pmatrix} a & -a\overline{z}^t J^{p,q} B & b \\ 0 & B & z \\ 0 & 0 & \overline{a}^{-1} \end{pmatrix} \;\middle|\; \begin{array}{l} a \in \mathbb{C}^*, b \in \mathbb{C}, B \in U(p,q), z \in \mathbb{C}^n, \\ a\overline{a}^{-1} det(B) = 1, \\ a^{-1} b + \overline{a}^{-1} \overline{b} + \langle z, z \rangle_{p,q} = 0 \end{array} \right\}.
$$

For $z = 0$ and $b = 0$ we obviously gain another subgroup

$$
G_0 := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & \overline{a}^{-1} \end{pmatrix} \;\middle|\; \begin{array}{l} a \in \mathbb{C}^*, B \in U(p,q) \\ a\overline{a}^{-1} det(B) = 1 \end{array} \right\}.
$$

In the next section we will see that the adjoint action of G_0 preserves the grading of the Lie algebra $\mathfrak{su}(p+1, q+1)$.

However before we do that we will identify another subgroup of the group of the special unitary transformations, the one with matrices of the shape $\begin{pmatrix} 1 & 0 & 0 \\ * & Id & 0 \\ * & * & 1 \end{pmatrix}$. Let us compute the possible entries.

$$
\begin{aligned}
\begin{pmatrix} 0 & 0 & 1 \\ 0 & J^{p,q} & 0 \\ 1 & 0 & 0 \end{pmatrix} &\overset{!}{=} A^* \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & J^{p,q} & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot A \\
&= A^* \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & J^{p,q} & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ y & Id & 0 \\ c & w^* & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & y^* & \overline{c} \\ 0 & Id & w \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c & w^* & 1 \\ J^{p,q}y & J^{p,q} & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} c + \overline{c} + \langle y, y\rangle_{p,q} & w^* + y^* J^{p,q} & 1 \\ w + J^{p,q} y & J^{p,q} & 0 \\ 1 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

Thus

$$
P_- := \left\{ A(y,c) := \begin{pmatrix} 1 & 0 & 0 \\ y & Id & 0 \\ ic - \frac{1}{2}\langle y, y\rangle_{p,q} & -y^* J^{p,q} & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} y \in \mathbb{C}^n \\ c \in \mathbb{R} \end{array} \right\}
$$

is a subset of the special unitary group $SU(p+1, q+1)$. This is actually an abelian subgroup $P_- \subset SU(p+1, q+1)$ due to $A(x,a) \cdot A(y,c) = A(x+y, a+c) = A(y,c) \cdot A(x,a) \in P_-$.

4.2 The Lie Algebra $\mathfrak{su}(1, n+1)$

Later on we will only consider strictly pseudo-convex CR manifolds. Hence it is sufficient to restrict to the signature $(1, n+1)$ in order to simplify the notation. Let us take a look at the Lie algebra $\mathfrak{g} := \mathfrak{su}(1, n+1)$. Again we use on $\mathbb{C}^{1,n+1}$ the hermitian form

$$
(x,y) \mapsto \langle x, y\rangle_S = x^* S y = \overline{x_{n+1}} y_0 + \sum_{j=1}^{n} \overline{x_j} y_j + \overline{x_0} y_{n+1},
$$

that is to say the matrix S is of the shape

$$
S = \begin{pmatrix} 0 & & 1 \\ & I_n & \\ 1 & & 0 \end{pmatrix},
$$

and the basis $\left(f_- := \frac{1}{\sqrt{2}}(e_0 - e_{n+1}), e_1, \ldots, e_n, f_+ := \frac{1}{\sqrt{2}}(e_0 + e_{n+1})\right)$ with f_- and f_+ being light like.

We want to explore $\mathfrak{g}$ in more detail. $\mathfrak{g}$ is the Lie algebra of $SU(1, n+1)$ and

$$
\begin{aligned}
SU(1, n+1) &= \left\{ A \in M(n+2, \mathbb{C}) \;\middle|\; \begin{array}{l} \langle Ax, Ay\rangle_S = \langle x, y\rangle_S \text{ for all } x, y \in \mathbb{C}^{n+2} \\ \text{and } Det(A) = 1 \end{array} \right\} \\
&= \{ A \in M(n+2, \mathbb{C}) \mid A^* \circ S \circ A = S \text{ and } Det(A) = 1 \}.
\end{aligned}
$$

Hence we get for the Lie algebra $\mathfrak{g}$:

$$
\begin{aligned}
\mathfrak{g} &= \mathfrak{su}(1, n+1) \\
&= \{ A \in M(n+2, \mathbb{C}) \mid A^* \circ S + S \circ A = 0 \text{ and } Tr(A) = 0 \}.
\end{aligned}
$$

We partition the matrices of $\mathfrak{g}$ into blocks of the sizes $1, n$ and 1:

$$\mathfrak{g} \ni B = \begin{pmatrix} c & Z^* & b \\ X & A & Y \\ a & W^* & d \end{pmatrix} \quad \text{with} \quad \begin{array}{l} a,b,c,d \in \mathbb{C}, \\ W,X,Y,Z \in \mathbb{C}^n \text{ and} \\ A \in M(n,\mathbb{C}). \end{array}$$

B is an element of $\mathfrak{g}$, if and only if $Tr(B) = 0$ and

$$\begin{aligned}
0 &\overset{!}{=} B^* \circ S + S \circ B \\
&= \begin{pmatrix} \overline{c} & X^* & \overline{a} \\ Z & A^* & W \\ \overline{b} & Y^* & \overline{d} \end{pmatrix} \circ \begin{pmatrix} 0 & & 1 \\ & I_n & \\ 1 & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & 1 \\ & I_n & \\ 1 & & 0 \end{pmatrix} \circ \begin{pmatrix} c & Z^* & b \\ X & A & Y \\ a & W^* & d \end{pmatrix} \\
&= \begin{pmatrix} \overline{a} & X^* & \overline{c} \\ W & A^* & Z \\ \overline{d} & Y^* & \overline{b} \end{pmatrix} + \begin{pmatrix} a & W^* & d \\ X & A & Y \\ c & Z^* & b \end{pmatrix} \\
&= \begin{pmatrix} \overline{a} + a & X^* + W^* & \overline{c} + d \\ W + X & A^* + A & Z + Y \\ \overline{d} + c & Y^* + Z^* & \overline{b} + b \end{pmatrix}.
\end{aligned}$$

Consequently B is an element of $\mathfrak{g}$, if and only if the real part of a and b vanishes, $d = -\overline{c}$, $W = -X$ and $Y = -Z$ are true, $A \in \mathfrak{u}(n)$ and $Tr(B) = Tr(A) + c - \overline{c} = 0$.

$$\mathfrak{g} = \left\{ \begin{pmatrix} z & -Z^* & ib \\ X & A & Z \\ ia & -X^* & -\overline{z} \end{pmatrix} \;\middle|\; \begin{array}{l} a,b \in \mathbb{R}, z \in \mathbb{C}, X, Z \in \mathbb{C}^n \\ A \in \mathfrak{u}(n) \text{ with } Tr(A) = \overline{z} - z \end{array} \right\}$$

We set:

$$\begin{aligned}
\mathfrak{g}_{-2} &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ ia & 0 & 0 \end{pmatrix} =: E_{-2}(a) \;\middle|\; a \in \mathbb{R} \right\}, \\
\mathfrak{g}_{-1} &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & -X^* & 0 \end{pmatrix} =: E_{-1}(X) \;\middle|\; X \in \mathbb{C}^n \right\}, \\
\mathfrak{g}_0 &= \left\{ \begin{pmatrix} z & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -\overline{z} \end{pmatrix} =: E_0(z,A) \;\middle|\; z \in \mathbb{C}, A \in \mathfrak{u}(n) \text{ with } Tr(A) = \overline{z} - z \right\}, \\
\mathfrak{g}_1 &= \left\{ \begin{pmatrix} 0 & -Z^* & 0 \\ 0 & 0 & Z \\ 0 & 0 & 0 \end{pmatrix} =: E_1(Z) \;\middle|\; Z \in \mathbb{C}^n \right\} \text{ and} \\
\mathfrak{g}_2 &= \left\{ \begin{pmatrix} 0 & 0 & ib \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =: E_2(b) \;\middle|\; b \in \mathbb{R} \right\}.
\end{aligned}$$

This leads to a 2-grading of $\mathfrak{g} = \mathfrak{su}(1, n+1)$. The Lie brackets are listed below.

$[\mathfrak{g}_{-2}, \mathfrak{g}_{-2}]$	$= 0$
$[\mathfrak{g}_{-2}, \mathfrak{g}_{-1}]$	$= 0$
$[\mathfrak{g}_{-2}, \mathfrak{g}_{0}]$	$[E_{-2}(a), E_0(z, A)] = E_{-2}\big(a(z + \overline{z})\big)$
$[\mathfrak{g}_{-2}, \mathfrak{g}_{1}]$	$[E_{-2}(a), E_1(Z)] = E_{-1}\big(- iaZ\big)$
$[\mathfrak{g}_{-2}, \mathfrak{g}_{2}]$	$[E_{-2}(a), E_2(b)] = E_0\big(ab, 0\big)$
$[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$	$[E_{-1}(X), E_{-1}(Y)] = E_{-2}\big(2Im(Y^*X)\big)$
$[\mathfrak{g}_{-1}, \mathfrak{g}_{0}]$	$[E_{-1}(X), E_0(z, A)] = E_{-1}\big(zX - AX\big)$
$[\mathfrak{g}_{-1}, \mathfrak{g}_{1}]$	$[E_{-1}(X), E_1(Z)] = E_0\big(Z^*X, ZX^* - XZ^*\big)$
$[\mathfrak{g}_{-1}, \mathfrak{g}_{2}]$	$[E_{-1}(X), E_2(b)] = E_1\big(ibX\big)$
$[\mathfrak{g}_{0}, \mathfrak{g}_{0}]$	$[E_0(z, A), E_0(w, B)] = E_0\big(0, [A, B]\big)$
$[\mathfrak{g}_{0}, \mathfrak{g}_{1}]$	$[E_0(z, A), E_1(Y)] = E_1\big(\overline{z}Y + AY\big)$
$[\mathfrak{g}_{0}, \mathfrak{g}_{2}]$	$[E_0(z, A), E_2(b)] = E_2\big(b(z + \overline{z})\big)$
$[\mathfrak{g}_{1}, \mathfrak{g}_{1}]$	$[E_1(Z), E_1(Y)] = E_2\big(2Im(Y^*Z)\big)$
$[\mathfrak{g}_{1}, \mathfrak{g}_{2}]$	$= 0$
$[\mathfrak{g}_{2}, \mathfrak{g}_{2}]$	$= 0$

According to Proposition 3.1 we have the grading element E, a uniquely defined element $E \in \mathfrak{g}_0$ satisfying $[E, X] = lX$ for all $X \in \mathfrak{g}_l$, $l = -2, \dots, 2$. From the corresponding commutators we find for the grading element $E = E_0(1, 0)$.
The Lie group $SU(1, n+1)$ acts via the Adjoint action on $\mathfrak{g}$, $Ad : SU(1, n+1) \longrightarrow Gl(\mathfrak{g})$. However this action is not faithful.
So we are looking for the centre $Z(G)$ of the group G, i.e. all matrices $A \in SU(1, n+1)$ such that for all $g \in \mathfrak{g}$ the equation $Ad(A)(g) = dL_A \circ dR_{A^{-1}} g = g$ holds. This is equivalent to $A \cdot g = g \cdot A$. Especially for $g = E_{-2}(a)$ and $g = E_{-1}(e_i)$ this leads to the following necessary condition for the matrix A: $A = x \cdot Id$. Furthermore the determinant of A has to be one. Hence we obtain $Ad(A) = Id$ if and only if $A = e^{\frac{2\pi i k}{n+2}} Id$, with $k = 0, 1, \dots, n+1$.
Therefore, we consider the group

$$\begin{aligned} \underline{G} &:= PSU(1, n+1) \\ &= SU(1, n+1)/_{Z(G)} \\ &= SU(1, n+1)/_{\{x \cdot Id \mid x \in \mathbb{C}, x^{2+n} = 1\}} \\ &= SU(1, n+1)/_{\mathbb{Z}_{n+2}}. \end{aligned}$$

The stabilizer of the complex line $\mathbb{C}f_-$ with respect to $\underline{G}$ is denoted by $\underline{P}$ and since the centre of G is actually a subgroup of P we have $\underline{P} = stab_{\underline{G}}(\mathbb{C}f_-) = stab_G(\mathbb{C}f_-)/_{Z(G)} = P/_{Z(G)}$.
To simplify calculations and notations we will most of the time neglect this fact and write $G = SU(1, n+1)$. However we will keep in mind that we actually work modulo $\mathbb{Z}_{n+2}$. The Lie algebra of the stabilizer of the complex line $\mathbb{C}f_-$ is the subalgebra $LA(P) = \mathfrak{p} = \mathfrak{g}_0 \oplus \dots \oplus \mathfrak{g}_k$.

The two definitions of the subgroup G_0,

$$\begin{aligned} G_0 &:= \{g \in G \mid Ad(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i,\ i = -2, \ldots, 2\} \\ \text{and } G_0 &:= \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & \overline{a}^{-1} \end{pmatrix} \;\middle|\; \begin{matrix} a \in \mathbb{C}^*, B \in U(p,q) \\ a\overline{a}^{-1} det(B) = 1 \end{matrix} \right\} \end{aligned}$$

are equivalent as we will check now.
We search for all matricies $g \in SU(1, n+1)$ (modulo $\mathbb{Z}_{n+2}$) for which for all $i = -2, \ldots, 2$ and all $K \in \mathfrak{g}_i$ it holds $g \cdot K \cdot g^{-1} \in \mathfrak{g}_i$. I.e. there exists a matrix $F \in \mathfrak{g}_i$ with $g \cdot K = F \cdot g$. We write

$$g = \begin{pmatrix} * & A & a \\ B & * & C \\ b & D & * \end{pmatrix}$$

and choose especially for $K = E_{-2}(x)$ and corresponding $F = E_{-2}(y)$. Then it has to hold:

$$g \cdot K = \begin{pmatrix} *iax & 0 & 0 \\ ixC & 0 & 0 \\ ix* & 0 & 0 \end{pmatrix} \stackrel{!}{=} F \cdot g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ iy* & iyA & iya \end{pmatrix}.$$

This results in the necessary conditions $a = 0$ and $A = C = 0$. If we choose $K = E_2(x)$ and $F = E_2(y)$, we obtain analogously $b = 0$ and $B = D = 0$. Furthermore for all matrices of $SU(1, n+1)$ we have $det(g) = 1$ and $g^* \cdot S \cdot g = S$. With $g = \begin{pmatrix} a & & \\ & B & \\ & & c \end{pmatrix}$ we get

$$g^* \cdot S \cdot g = \begin{pmatrix} 0 & & \overline{a}c \\ & B^*B & \\ a\overline{c} & & 0 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 & & 1 \\ & I_n & \\ 1 & & 0 \end{pmatrix}.$$

I.e. $B \in U(n)$, $c = \frac{a}{|a|^2}$ and $det(g) = \frac{a^2}{|a|^2} det(B) = 1$. We conclude:

$$\begin{aligned} G_0 &:= \left\{ g \in SU(1, n+1) \;\middle|\; Ad(g)\big(\mathfrak{g}_i\big) \subset \mathfrak{g}_i,\ i = -2, \ldots, 2 \right\} \\ &= \left\{ \begin{pmatrix} a & & 0 \\ & B & \\ 0 & & \frac{a}{|a|^2} \end{pmatrix} \;\middle|\; a \in \mathbb{C}, B \in U(n), \frac{a^2}{|a|^2} det(B) = 1 \right\}. \end{aligned}$$

We will now compute the Adjoint action of G_0 on $\mathfrak{g}_- := \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ for later use.
Let $(\varphi, \phi) := \begin{pmatrix} \varphi & & \\ & \phi & \\ & & \frac{\varphi}{|\varphi|^2} \end{pmatrix} \in G_0$. Then we have

$$\begin{aligned} Ad((\varphi, \phi))(E_{-2}(a) \oplus E_{-1}(X)) &= \begin{pmatrix} \varphi & & \\ & \phi & \\ & & \frac{\varphi}{|\varphi|^2} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ ia & -X^* & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\overline{\varphi}}{|\varphi|^2} & & \\ & \phi^* & \\ & & \overline{\varphi} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ \frac{\overline{\varphi}}{|\varphi|^2}\phi X & 0 & 0 \\ i\frac{1}{|\varphi|^2}a & -\frac{\varphi}{|\varphi|^2}X^*\phi^* & 0 \end{pmatrix} \\ &= E_{-2}\left(\frac{a}{|\varphi|^2}\right) \oplus E_{-1}\left(\varphi^{-1}\phi X\right). \end{aligned}$$

4.2.1 The Killing Form of $\mathfrak{su}(1, n+1)$, $\mathfrak{g}_2^*$, $\mathfrak{g}_1^*$ and σ

The Killing form of $\mathfrak{su}(1, n+1)$ can be computed with the help of the Killing form of $\mathfrak{gl}(n+2, \mathbb{C})$ since $\mathfrak{su}(1, n+1)$ is an ideal of $\mathfrak{u}(1, n+1)$ whose complexification is $\mathfrak{gl}(n+2, \mathbb{C})$. So we have for the Killing forms

$$B_{\mathfrak{su}(1,n+1)} = B_{\mathfrak{u}(1,n+1)}|_{\mathfrak{su}(1,n+1)\times\mathfrak{su}(1,n+1)} = B_{\mathfrak{gl}(n+2,\mathbb{C})}|_{\mathfrak{su}(1,n+1)\times\mathfrak{su}(1,n+1)}.$$

Thus it is sufficient to compute the Killing form of $\mathfrak{gl}(n+2, \mathbb{C})$. With the help of the map $\mathfrak{gl}(n+2, \mathbb{C}) \longrightarrow (\mathbb{C}^{n+2})^* \otimes \mathbb{C}^{n+2}$ defined by linear prolongation via $Z \cdot W^t \mapsto W^* \otimes Z$, we find that the adjoint action of $\mathfrak{gl}(n+2, \mathbb{C})$ on itself corresponds to the following action of $\mathfrak{gl}(n+2, \mathbb{C})$ on $(\mathbb{C}^n)^* \otimes \mathbb{C}^n$

$$\begin{array}{rcl} \varrho(X) : (\mathbb{C}^n)^* \otimes \mathbb{C}^n & \longrightarrow & (\mathbb{C}^n)^* \otimes \mathbb{C}^n \\ W^* \otimes Z & \mapsto & W^* \otimes (XZ) - (X^t W)^* \otimes Z. \end{array}$$

Hence we can write for the Killing form

$$\begin{array}{rcl} B_{\mathfrak{gl}(n+2,\mathbb{C})}(X,Y) & = & Tr\big(ad(X) \circ ad(Y)\big) \\ & = & \sum_{i,k} \big\langle \varrho(X) \circ \varrho(Y) e_i^* \otimes e_k, e_i^* \otimes e_k \big\rangle \\ & = & \sum_{i,k} \Big\langle \varrho(X)\big(e_i^* \otimes Y e_k - (Y^t e_i)^* \otimes e_k\big), e_i^* \otimes e_k \Big\rangle \\ & = & \sum_{i,k} \Big\langle e_i^* \otimes XY e_k - (Y^t e_i)^* \otimes X e_k \\ & & \qquad -(X^t e_i)^* \otimes Y e_k + (X^t Y^t e_i)^* \otimes e_k, e_i^* \otimes e_k \Big\rangle \\ & = & \sum_{i,k} \langle XY e_k, e_k \rangle - \sum_{i,k} \langle Y^t e_i, e_i \rangle \cdot \langle X e_k, e_k \rangle \\ & & -\sum_{i,k} \langle X^t e_i, e_i \rangle \cdot \langle Y e_k e_k \rangle + \sum_{i,k} \langle X^t Y^t e_i, e_i \rangle \\ & = & 2(n+2) Tr(X \cdot Y) - 2Tr(X)Tr(Y). \end{array}$$

Restriction to $\mathfrak{su}(1, n+1)$ leads with $Tr(X) = 0$ to

$$B_{\mathfrak{su}(1,n+1)}(X,Y) = 2(n+2)Tr(X \cdot Y).$$

The dual of $\mathfrak{g}_2$ and $\mathfrak{g}_1$ is given by the Killing form $B_{\mathfrak{su}(1,n+1)}$:

$$\begin{array}{rcl} E_2(a)^* & = & E_{-2}\left(-\frac{1}{2(n+2)a}\right), \\ E_1(X)^* & = & E_{-1}\left(-\frac{1}{4(n+2)\|X\|^2} X\right), \\ E_{-1}(X)^* & = & E_1\left(-\frac{1}{4(n+2)\|X\|^2} X\right) \text{ and} \\ E_{-2}(a)^* & = & E_2\left(-\frac{1}{2(n+2)a}\right). \end{array}$$

Furthermore we can now determine the involutive linear automorphism $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$ which satisfies $\sigma(\mathfrak{g}_i) = \mathfrak{g}_{-i}$ and $B_{\mathfrak{su}(1,n+1)}\big(X, \sigma(X)\big) < 0$ for all non-zero elements X in $\mathfrak{g}$. We have

$$\begin{array}{rcl} \sigma\big(E_{-2}(a)\big) & = & E_2(a), \\ \sigma\big(E_{-1}(X)\big) & = & E_1(X) \text{ and} \\ \sigma\big(E_0(z, A)\big) & = & E_0(-\overline{z}, -A^*). \end{array}$$

4.2.2 The Action of P on $\mathfrak{g}_i$ and $(\mathfrak{g}_i)^*$

The subgroup $P \subset G$ is given by

$$\begin{array}{rcl} P & = & \{p \in G \mid Ad(p)(\mathfrak{g}^j) \subset \mathfrak{g}^j \text{ for all } j = -2, \dots, 2\} \\ & = & \left\{ A = \begin{pmatrix} a & -aZ^*B & b \\ 0 & B & Z \\ 0 & 0 & \overline{a}^{-1} \end{pmatrix} \;\middle|\; \begin{array}{l} a \in \mathbb{C}^*, b \in \mathbb{C}, B \in U(n), Z \in \mathbb{C}^n \\ a\overline{a}^{-1} det(B) = 1 \\ a^{-1}b + \overline{a}^{-1}\overline{b} + \|Z\|^2 = 0 \end{array} \right\}. \end{array}$$

Its Lie algebra is $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$. The inverse of a matrix $A \in P$ is

$$A^{-1} = \begin{pmatrix} a^{-1} & Z^* & \bar{b} \\ 0 & B^* & -\bar{a}B^*Z \\ 0 & 0 & \bar{a} \end{pmatrix}.$$

Please recall the notation $\mathfrak{g}^j := \mathfrak{g}_j \oplus \mathfrak{g}_{j+1} \oplus \cdots \oplus \mathfrak{g}_k$. We will now compute the adjoint action of $A \in P$ on elements $E_{-2}(c) + \mathfrak{g}^{-1}$ and $E_{-1}(X) + \mathfrak{p}$. We get

$$\begin{aligned}
Ad(A)\big(E_{-2}(c) + \mathfrak{g}^{-1}\big) &= A \cdot E_{-2}(c) \cdot A^{-1} + \mathfrak{g}^{-1} \\
&= \begin{pmatrix} a & -aZ^*B & b \\ 0 & B & Z \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ ic & 0 & 0 \end{pmatrix} \cdot A^{-1} + \mathfrak{g}^{-1} \\
&= \begin{pmatrix} ibc & 0 & 0 \\ icZ & 0 & 0 \\ ic\bar{a}^{-1} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a^{-1} & Z^* & \bar{b} \\ 0 & B^* & -\bar{a}B^*Z \\ 0 & 0 & \bar{a} \end{pmatrix} + \mathfrak{g}^{-1} \\
&= \begin{pmatrix} * & & \\ ia^{-1}cZ & * & \\ i\frac{c}{|a|^2} & -(ia^{-1}cZ)^* & * \end{pmatrix} \\
&= E_{-2}\big(\tfrac{c}{|a|^2}\big) + \mathfrak{g}^{-1}
\end{aligned}$$

and

$$\begin{aligned}
Ad(A)\big(E_{-1}(X) + \mathfrak{p}\big) &= A \cdot E_{-1}(X) \cdot A^{-1} + \mathfrak{p} \\
&= \begin{pmatrix} a & -aZ^*B & b \\ 0 & B & Z \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & -X^* & 0 \end{pmatrix} \cdot A^{-1} + \mathfrak{p} \\
&= \begin{pmatrix} -aZ^*BX & -bX^* & 0 \\ BX & -ZX^* & 0 \\ 0 & -\bar{a}^{-1}X^* & 0 \end{pmatrix} \cdot \begin{pmatrix} a^{-1} & Z^* & \bar{b} \\ 0 & B^* & -\bar{a}B^*Z \\ 0 & 0 & \bar{a} \end{pmatrix} + \mathfrak{p} \\
&= \begin{pmatrix} * & & \\ a^{-1}BX & * & \\ 0 & -(a^{-1}BX)^* & * \end{pmatrix} + \mathfrak{p} \\
&= E_{-1}(a^{-1}BX) + \mathfrak{p}.
\end{aligned}$$

In the same way we obtain for $E_2(c)$ and $E_1(X) + \mathfrak{g}^2$:

$$\begin{aligned}
Ad(A)E_2(c) &= A \cdot E_2(c) \cdot A^{-1} \\
&= \begin{pmatrix} a & -aZ^*B & b \\ 0 & B & Z \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & ic \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot A^{-1} \\
&= \begin{pmatrix} 0 & 0 & iac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a^{-1} & Z^* & \bar{b} \\ 0 & B^* & -\bar{a}B^*Z \\ 0 & 0 & \bar{a} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & i|a|^2c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= E_2(|a|^2c)
\end{aligned}$$

and

$$
\begin{aligned}
Ad(A)E_1(X)+\mathfrak{g}^2 &= A\cdot E_1(X)\cdot A^{-1}+\mathfrak{g}^2\\
&= \begin{pmatrix} a & -aZ^*B & b\\ 0 & B & Z\\ 0 & 0 & \overline{a}^{-1}\end{pmatrix}\cdot\begin{pmatrix} 0 & -X^* & 0\\ 0 & 0 & X\\ 0 & 0 & 0\end{pmatrix}\cdot A^{-1}+\mathfrak{g}^2\\
&= \begin{pmatrix} 0 & -aX^* & -aZ^*BX\\ 0 & 0 & BX\\ 0 & 0 & 0\end{pmatrix}\cdot\begin{pmatrix} a^{-1} & Z^* & \overline{b}\\ 0 & B^* & -\overline{a}B^*Z\\ 0 & 0 & \overline{a}\end{pmatrix}+\mathfrak{g}^2\\
&= \begin{pmatrix} 0 & -aX^*B^* & |a|^2\big(X^*B^*Z-Z^*BX\big)\\ 0 & 0 & \overline{a}BX\\ 0 & 0 & 0\end{pmatrix}+\mathfrak{g}^2\\
&= E_1(\overline{a}BX)+\mathfrak{g}^2.
\end{aligned}
$$

So we have the following actions of P on $\mathfrak{g}_{-2},\ldots,\mathfrak{g}_2$:

$$
\begin{aligned}
Ad_{\mathfrak{g}_{-2}}:P\times\mathfrak{g}_{-2} &\longrightarrow \mathfrak{g}_{-2}\\
\big(A,E_{-2}(c)\big) &\mapsto Ad_{\mathfrak{g}_{-2}}(A)E_{-2}(c)=pr_{\mathfrak{g}_{-2}}\circ Ad(A)E_{-2}(c)=E_{-2}(\tfrac{c}{|a|^2}),
\end{aligned}
$$

$$
\begin{aligned}
Ad_{\mathfrak{g}_{-1}}:P\times\mathfrak{g}_{-1} &\longrightarrow \mathfrak{g}_{-1}\\
\big(A,E_{-1}(X)\big) &\mapsto Ad_{\mathfrak{g}_{-1}}(A)E_{-1}(X)=pr_{\mathfrak{g}_{-1}}\circ Ad(A)E_{-1}(X)=E_{-1}(a^{-1}BX),
\end{aligned}
$$

$$
\begin{aligned}
Ad_{\mathfrak{g}_{1}}:P\times\mathfrak{g}_{1} &\longrightarrow \mathfrak{g}_{1}\\
\big(A,E_{1}(X)\big) &\mapsto Ad_{\mathfrak{g}_{1}}(A)E_{1}(X)=pr_{\mathfrak{g}_{1}}\circ Ad(A)E_{1}(X)=E_{1}(\overline{a}BX)\text{ and}
\end{aligned}
$$

$$
\begin{aligned}
Ad_{\mathfrak{g}_{2}}:P\times\mathfrak{g}_{2} &\longrightarrow \mathfrak{g}_{2}\\
\big(A,E_{2}(c)\big) &\mapsto Ad_{\mathfrak{g}_{2}}(A)E_{2}(c)=E_{2}(|a|^2c).
\end{aligned}
$$

The dualisation $\mathfrak{g}^*_{-j}\simeq\mathfrak{g}_j$ for $j=1,2$ is given by the Killing form. So we obtain with $Ad^*(A)X^*=(Ad(A)X)^*$ for the dual action of P on $\mathfrak{g}^*_{-2}$ and $\mathfrak{g}^*_{-1}$

$$
\begin{aligned}
Ad^*_{\mathfrak{g}_{-2}} &= Ad_{\mathfrak{g}_2},\\
Ad^*_{\mathfrak{g}_{-1}} &= Ad_{\mathfrak{g}_1}.
\end{aligned}
$$

Later we will describe the canonical complex line bundle of a CR-manifold as a bundle associated to the Cartan bundle. For this purpose we set $\mathfrak{g}_{10}:=\{X-iJX\mid X\in\mathfrak{g}_{-1}\}\subset\mathfrak{g}^{\mathbb{C}}_{-1}$ analogously to the recovery of the complex subbundle T_{10} from the real subbundle H. Here J denotes the complex structure on $\mathfrak{g}_{-1}$ and i the one coming from the complexification. Hence a basis of $\mathfrak{g}_{10}$ is given by

$$\big(E_{-1}(e_1)-iE_{-1}(Je_1),\ldots,E_{-1}(e_n)-iE_{-1}(Je_n)\big).$$

Further we define $\mathfrak{g}^{\mathbb{C}}_{-2}\wedge\mathfrak{g}_{10}^{\wedge n}:=\mathfrak{g}^{\mathbb{C}}_{-2}\wedge\underbrace{\mathfrak{g}_{10}\wedge\cdots\wedge\mathfrak{g}_{10}}_{n\text{ times}}$.

A basis of $\mathfrak{g}^{\mathbb{C}}_{-2}\wedge\mathfrak{g}_{10}^{\wedge n}$ is given by

$$\eta := E_{-2}(1)\wedge\big(E_{-1}(e_1)-iE_{-1}(Je_1)\big)\wedge\cdots\wedge\big(E_{-1}(e_n)-iE_{-1}(Je_n)\big).$$

Now we can explain the action of P on $\mathfrak{g}^{\mathbb{C}}_{-2}\wedge\mathfrak{g}_{10}^{\wedge n}$ and its dual which we will denote by $\widetilde{Ad}^*$ and $\widetilde{Ad}$.

For an element $A = \begin{pmatrix} a & -aZ^*B & b \\ 0 & B & Z \\ 0 & 0 & \overline{a}^{-1} \end{pmatrix} \in P$ we have

$$\begin{aligned}
\widetilde{Ad}^*(A)(\eta) &= Ad^{\mathbb{C}}_{\mathfrak{g}_{-2}}(A)\big(E_{-2}(1)\big) \\
&\quad \wedge \big(Ad_{\mathfrak{g}_{-1}}(A)E_{-1}(e_1) - iAd_{\mathfrak{g}_{-1}}(A)E_{-1}(Je_1)\big) \\
&\quad \wedge \cdots \wedge \big(Ad_{\mathfrak{g}_{-1}}(A)E_{-1}(e_n) - iAd_{\mathfrak{g}_{-1}}(A)E_{-1}(Je_n)\big) \\
&= E_{-2}(\tfrac{1}{|a|^2}) \\
&\quad \wedge\big(E_{-1}(a^{-1}Be_1) - iE_{-1}(a^{-1}BJe_1)\big) \\
&\quad \wedge \cdots \wedge \big(E_{-1}(a^{-1}Be_n) - iE_{-1}(a^{-1}BJe_n)\big) \\
&= \tfrac{det(B)}{|a|^2 a^n}\eta.
\end{aligned}$$

Since we have $a\overline{a}^{-1}det(B) = 1$ we can conclude

$$\widetilde{Ad}^*(A) = a^{-(n+2)} : \mathfrak{g}^{\mathbb{C}}_{-2} \wedge \mathfrak{g}^{\wedge n}_{10} \longrightarrow \mathfrak{g}^{\mathbb{C}}_{-2} \wedge \mathfrak{g}^{\wedge n}_{10}.$$

We denote this action with $\widetilde{Ad}^*$ since it corresponds to the action of P on the dual of the canonical line bundle, $\mathcal{K} = \Big\{\omega \in \bigwedge^{n+1}(TM^{\mathbb{C}})^* \,\Big|\, i_V\omega = 0 \text{ for all } V \in \overline{T_{10}}\Big\}$. Accordingly the action on the dual space $\big(\mathfrak{g}^{\mathbb{C}}_{-2} \wedge \mathfrak{g}^{\wedge n}_{10}\big)^* = \mathfrak{g}^{\mathbb{C}}_2 \wedge \big(\mathfrak{g}^{\wedge n}_{10}\big)^*$ is denoted by $\widetilde{Ad}$.

$$\begin{aligned}
\widetilde{Ad}(A)(\hat{E}) &= Ad^{\mathbb{C}}_{\mathfrak{g}_2}(A)\big(E_2(1)\big) \\
&\quad \wedge (Ad_{\mathfrak{g}_1}(A)E_1(e_1) - iAd_{\mathfrak{g}_1}(A)E_1(Je_1)) \\
&\quad \wedge \cdots \wedge (Ad_{\mathfrak{g}_1}(A)E_1(e_n) - iAd_{\mathfrak{g}_1}(A)E_1(Je_n)) \\
&= E_2(|a|^2) \\
&\quad \wedge\big(E_1(\overline{a}Be_1) - iE_1(\overline{a}BJe_1)\big) \\
&\quad \wedge \cdots \wedge \big(E_1(\overline{a}Be_n) - iE_1(\overline{a}BJe_n)\big) \\
&= det(B)|a|^2\overline{a}^n\hat{E} \qquad\qquad \text{note: } det(B) = \tfrac{\overline{a}}{a} \\
&= \overline{a}^{n+2}\hat{E}
\end{aligned}$$

I.e. we have

$$\widetilde{Ad}(A) = \overline{a}^{n+2} : \mathfrak{g}^{\mathbb{C}}_2 \wedge (\mathfrak{g}^*_{10})^{\wedge n} \longrightarrow \mathfrak{g}^{\mathbb{C}}_2 \wedge (\mathfrak{g}^*_{10})^{\wedge n},$$

which corresponds to the action of P on the canonical line bundle $\mathcal{K}$.

4.2.3 The Cohomology Group

As we have seen one requirement for the prolongation procedure are the trivial cohomology groups $H^1_l(\mathfrak{g}_-, \mathfrak{g})$ for $l > 0$. This poses no problem for CR-geometries as can be seen from the following two propositions from [CS00] Sections 2.7 and 2.8.

Proposition 4.1 *Let $\mathfrak{g}$ be a complex simple $|k|$-graded Lie algebra. Then for each $l > 0$ the cohomology group $H^1_l(\mathfrak{g}_-, \mathfrak{g})$ is trivial, except in the following cases (using the Dynkin diagram notation - the crosses denote the simple roots contained in Σ):*

1. $\times$, *i.e. $\mathfrak{g} = A_1 = \mathfrak{sl}(2, \mathbb{C})$, and $\mathfrak{p} \subset \mathfrak{g}$ is the Borel subalgebra. In this case, $H^1_2(\mathfrak{g}_-, \mathfrak{g})$ is the only nonzero component with $l > 0$.*

2. $\times\!-\!\bullet\!-\cdots-\!\bullet\!-\!\bullet \simeq \bullet\!-\!\bullet\!-\cdots-\!\bullet\!-\!\times$, *i.e. $\mathfrak{g} = A_n = \mathfrak{sl}(n + 1, \mathbb{C})$ for some $n > 1$, and $\mathfrak{p}$ is the maximal parabolic corresponding to either the first or the last root. In this case, $H^1_1(\mathfrak{g}_-, \mathfrak{g})$ is the nonzero component.*

3. $\times\!-\!\bullet\!-\cdots-\!\bullet\!\Leftarrow\!\bullet$, *i.e. $\mathfrak{g} = C_n = \mathfrak{sp}(2n, \mathbb{C})$ for some $n \geq 2$, and $\mathfrak{p}$ is the maximal parabolic corresponding to the first root. In this case, $H^1_1(\mathfrak{g}_-, \mathfrak{g})$ is the nonzero component.*

In the real simple case the first cohomology group of positive homogeneity of a complexification is the complexification of the corresponding real cohomology group of the same homogeneity according to Lemma 3.5. of [Yam93]. Now the semisimple case can be deduced from the simple one using the following proposition from [CS00] Section 2.8.

Proposition 4.2 *Let $\mathfrak{g}'$ be a semisimple $|k'|$-graded Lie algebra such that no simple factor is contained in $\mathfrak{g}'_0$ and $\mathfrak{g}''$ be a semisimple $|k''|$-graded Lie algebra such that no simple factor is contained in $\mathfrak{g}''_0$, and put $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$. Then for each $l > 0$ we have*

$$H^1_l(\mathfrak{g}_-, \mathfrak{g}) \simeq H^1_l(\mathfrak{g}'_-, \mathfrak{g}') \oplus H^1_l(\mathfrak{g}''_-, \mathfrak{g}'').$$

If $k', k'' \geq 2$, then the result also holds for $l = 0$.

4.3 Preparations for the Prolongation Procedure

In this section, based on [CS00], we want to identify the structures needed for the prolongation procedure discussed in Section 3.5. Let M^{2n+1} be a smooth real manifold with a subbundle $T^{-1}M \subset TM$ of dimension $dimT^{-1}M = 2n$. By setting $T^{-2}M := TM$ and $T^0M := M \times \{0\}$ we obtain the filtration $T^{-2}M \supset T^{-1}M \supset T^0M$. We continue using the notations from the sections above, $G = SU(1, n+1)$ with Lie algebra $\mathfrak{g} = \mathfrak{su}(1, n+1)$, $P = Stab(\mathbb{C}f_-)$, $G_0 = \{g \in G \mid Ad(g)$ respects the grading $\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k\}$ and $P_+ = \exp(\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k)$ keeping in mind, that we actually work modulo $\mathbb{Z}_{n+2}$. Furthermore let $p : E \longrightarrow M$ be a G_0-principal bundle ($G_0 = P/P_+ = P/P^1_+$) and $\theta = (\theta_{-2}, \theta_{-1})$ a frame form of length one on E, i.e.

- θ_j is a smooth section of $L(\underbrace{T^jE}_{dp^{-1}(T^jM)}, \mathfrak{g}_j)$, in more detail:
  - θ_{-1} is a smooth section of $L(T^{-1}E, \mathfrak{g}_{-1})$ with $T^{-1}E = dp^{-1}(T^{-1}M)$,
  - with $T^{-2}E = dp^{-1}(T^{-2}M) = dp^{-1}(TM) = TE$ the form θ_{-2} is a smooth section of $L(TE, \mathfrak{g}_{-2})$,
- $Ker\big(\theta_j|_u\big) = T^{j+1}_uE = dp^{-1}(T^{j+1}_{p(u)}M)$ for all $u \in E$,
  - $Ker\big(\theta_{-1}|_u\big) = T^0_uE = dp^{-1}(T^0_{p(u)}M) = Tv_uE$ for all $u \in E$,
  - $Ker\big(\theta_{-2}|_u\big) = T^{-1}_uE = dp^{-1}(T^{-1}_{p(u)}M) \supset Tv_uE$ for all $u \in E$,
- θ_j is G_0-equivariant, that is $R^*_b\theta_j = Ad(b^{-1}) \circ \theta_j$ for all $b \in G_0$.

For every $u \in E$ the section θ_{-1} induces a linear isomorphism

$$\begin{array}{rcl} T^{-1}_xM \simeq T^{-1}_uE/_{T^0_uE} & \longrightarrow & \mathfrak{g}_{-1} \\ X & \mapsto & \theta_{-1}(dp^{-1}X) \end{array}$$

and θ_{-2} induces a linear isomorphism

$$\begin{array}{rcl} T_xM/_{T^{-1}_xM} \simeq T^{-2}_uE/_{T^{-1}_uE} & \longrightarrow & \mathfrak{g}_{-2} \\ [X] & \mapsto & \theta_{-2}(dp^{-1}X). \end{array}$$

Especially we obtain from the smooth section θ_{-1} a complex structure J on T^{-1}_xM defined by $JX := dp \circ \theta^{-1}_{-1}\big(i \cdot \theta_{-1} \circ dp^{-1}(X)\big)$. This definition is independent of u, since the G_0-action on $\mathfrak{g}_{-1}$ preserves the complex structure.
The structure function of degree -2 is:

$$d\theta_{-2}(\cdot, \cdot) + [\theta_{-1}, \theta_{-1}] : T^{-1}_uE \times T^{-1}_uE \longrightarrow \mathfrak{g}_{-2}.$$

Definition 4.1 *Let θ be a frame form of length one. θ satisfies the structure equations if and only if the structure function of degree -2 vanishes.*

The Lie-bracket $[\cdot,\cdot] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \longrightarrow \mathfrak{g}_{-2}$ is preserved under the Adjoint action of G_0 and can therefore be pulled back. We obtain a bilinear skew symmetric map between the bundles:

$$\begin{array}{rcl} \{\cdot,\cdot\} : T^{-1}M \times T^{-1}M & \longrightarrow & TM/T^{-1}M \\ (\xi_x, \eta_x) & \mapsto & \{\xi_x, \eta_x\} := dp \circ \theta_{-2}^{-1}\Big(\big[\theta_{-1} \circ dp^{-1}(\xi_x), \theta_{-1} \circ dp^{-1}(\eta_x)\big]_{\mathfrak{g}}\Big) \\ & & \text{for all } x \in M \text{ and all } \xi_x, \eta_x \in T_x^{-1}M. \end{array}$$

Lemma 4.1 *$\{\cdot,\cdot\}$ is nondegenerate and totally real, that is to say $\{J\xi, J\eta\} = \{\xi, \eta\}$ for all vectors $\xi, \eta \in T_x^{-1}M$.*

Proof: To see that the bilinear pairing $\{\cdot,\cdot\}$ is truly nondegenerate let ξ be a nonzero vector in $T_x^{-1}M$. Then $\theta_{-1} \circ dp^{-1}(\xi)$ is also nonzero since $\theta_{-1} \circ dp^{-1}$ is a linear isomorphism. Recall that we have for the Lie bracket $[E_{-1}(X), E_{-1}(Y)]_{\mathfrak{g}} = E_{-2}(2Im(Y^*X))$ (see Section 4.2). Thus we obtain

$$\begin{array}{rcl} \{\xi, J\xi\} & = & dp \circ \theta_{-2}^{-1}\big([\theta_{-1} \circ dp^{-1}(\xi), \theta_{-1} \circ dp^{-1}(J\xi)]_{\mathfrak{g}}\big) \\ & = & dp \circ \theta_{-2}^{-1}\big(\underbrace{[\theta_{-1} \circ dp^{-1}(\xi), i\theta_{-1} \circ dp^{-1}(\xi)]_{\mathfrak{g}}}_{\neq 0}\big) \\ & \neq & 0. \end{array}$$

I.e. $\{\cdot,\cdot\} : T^{-1}M \times T^{-1}M \longrightarrow TM/_{T^{-1}M}$ is nondegenerate.
Now we prove that the pairing is also totally real. According to Section 4.2 we have for the Lie bracket of elements of $\mathfrak{g}_{-1}$

$$[iX, iY]_{\mathfrak{g}} = [X, Y]_{\mathfrak{g}}.$$

Thus the claimed property follows directly.

$$\begin{array}{rcl} \{J\xi, J\eta\} & = & dp \circ \theta_{-2}^{-1}\big([\theta_{-1} \circ dp^{-1}(J\xi_x), \theta_{-1} \circ dp^{-1}(J\eta_x)]_{\mathfrak{g}}\big) \\ & = & dp \circ \theta_{-2}^{-1}\big([i\theta_{-1} \circ dp^{-1}(\xi_x), i\theta_{-1} \circ dp^{-1}(\eta_x)]_{\mathfrak{g}}\big) \\ & = & dp \circ \theta_{-2}^{-1}\big([\theta_{-1} \circ dp^{-1}(\xi_x), \theta_{-1} \circ dp^{-1}(\eta_x)]_{\mathfrak{g}}\big) \\ & = & \{\xi, \eta\} \end{array}$$

$\square$

The other way round assume we have given a smooth odd dimensional manifold M^{2n+1} with a complex subbundle $T^{-1}M$ of TM of complex rank $rank_{\mathbb{C}}T^{-1}M = n$ and a bilinear pairing $\{\cdot,\cdot\} : T^{-1}M \times T^{-1}M \longrightarrow TM/T^{-1}M$ which is nondegenerate in every point and totally real, i.e. $\{J\xi, J\eta\} = \{\xi, \eta\}$ for all $\xi, \eta \in T_x^{-1}M$. For each point $x \in M$ we have an isomorphism $T_xM/_{T_x^{-1}M} \simeq \mathbb{R}$ so that

$$\{\cdot,\cdot\} : T_x^{-1}M \times T_x^{-1}M \longrightarrow \mathbb{R}$$

is the imaginary part of a hermitian form $h(X, Y) = \{X, JY\} + i\{X, Y\}$.
Assume that the hermitian form h is positive definite (with an appropriate isomorphism $T_xM/_{T_x^{-1}M} \xrightarrow{\sim} \mathbb{R}$). I.e. this fixes an orientation of the line bundle $TM/_{T^{-1}M}$. We set E to be the following subbundle of the frame bundle of TM:

$$E := \left\{ (\varphi_1, \varphi_2)_x \;\middle|\; \begin{array}{c} x \in M,\ \varphi_1 : \mathfrak{g}_{-1} \longrightarrow T_x^{-1}M \text{ complex-linear isomorphism,} \\ \varphi_2 : \mathfrak{g}_{-2} \longrightarrow T_xM/T_x^{-1}M \text{ linear isomorphism with} \\ \{\varphi_1(X), \varphi_1(Y)\} = \varphi_2[X, Y] \text{ for all } X, Y \in \mathfrak{g}_{-1} \end{array} \right\}.$$

With $p : E \longrightarrow M$ we denote the obvious projection.

Lemma 4.2 *$p : E \longrightarrow M$ is a G_0-principal bundle, where G_0 acts by composition with the Adjoint action from the right,*

$$R_b(\varphi_1, \varphi_2) := (\varphi_1 \circ Ad(b), \varphi_2 \circ Ad(b)) \text{ for all } b \in G_0.$$

Proof: The action of G_0 on the bundle E is well defined since the Adjoint action of G_0 on the Lie algebra $\mathfrak{g}$ preserves the grading and we have

$$\begin{aligned} \{R_b\varphi_1(X), R_b\varphi_1(Y)\} &= \{\varphi_1\big(Ad(b)(X)\big), \varphi_1\big(Ad(b)(Y)\big)\} \\ &= \varphi_2\big([Ad(b)(X), Ad(b)(Y)]_{\mathfrak{g}}\big) \\ &= \varphi_2 \circ Ad(b)\big([X,Y]_{\mathfrak{g}}\big) \\ &= (R_b\varphi_2)([X,Y]_{\mathfrak{g}}). \end{aligned}$$

I.e. $R_b(\varphi_1, \varphi_2)$ is an element of E for every $b \in G_0$ and $(\varphi_1, \varphi_2) \in E$. The action is free since we work modulo the center of G and with φ_1 and φ_2 being isomorphisms the equation $R_b(\varphi_1, \varphi_2) = (\varphi_1, \varphi_2)$ is only true for $Ad(b)$ being the identity which implies that b is the neutral element. To see that the action of G_0 is transitive on the fibres let (φ_1, φ_2) and (ψ_1, ψ_2) be two elements in the same fibre of E. Then

$$(\varphi_2^{-1} \circ \psi_2) \oplus (\varphi_1^{-1} \circ \psi_1) : \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \longrightarrow \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$$

is an automorphism of $\mathfrak{g}_-$ which preserves the grading. Thus we have a $p \in G_0$ with $Ad(p) = (\varphi_1^{-1} \circ \psi_1, \varphi_2^{-1} \circ \psi_2)$ and so $(\psi_1, \psi_2) = R_p(\varphi_1, \varphi_2)$.
Therefore, $p : E \longrightarrow M$ is a G_0-principal bundle.

□

We define a frame form θ on E by:

$$\begin{aligned} \theta_{-2}(\varphi)(\xi) &:= \varphi_2^{-1}([dp\xi]) && \text{for all } \xi \in T_\varphi E = T_\varphi^{-2}E, \\ \theta_{-1}(\varphi)(\xi) &:= \varphi_1^{-1}(\underbrace{dp\xi}_{\in T_x^{-1}M}) && \text{for all } \xi \in dp^{-1}(T_x^{-1}M) = T_\varphi^{-1}E. \end{aligned}$$

With $\varphi_2 : \mathfrak{g}_{-2} \longrightarrow T_xM/_{T_x^{-1}M}$ being an isomorphism the kernel of θ_{-2} is $T^{-1}E$ and with $\varphi_1 : \mathfrak{g}_{-1} \longrightarrow T_x^{-1}M$ being an isomorphism the kernel of θ_{-1} is the vertical subbundle T^0E. θ_{-2} and θ_{-1} are also G_0-equivariant since according to the definition of the right action of G_0 on E we have

$$\begin{aligned} (R_b^*\theta_j)(\varphi) &= \theta_j(R_b\varphi) \\ &= \theta_j(\varphi \circ Ad(b)) \\ &= \big(\varphi_j \circ Ad(b)\big)^{-1} \\ &= Ad(b^{-1}) \circ \varphi_j^{-1} \\ &= Ad(b^{-1}) \circ \theta_j(\varphi). \end{aligned}$$

Thus $(\theta_{-2}, \theta_{-1})$ is a frame form of length one.

Proposition 4.3 *We have a bijection*

$$\left\{ \begin{array}{c} G_0\text{-principal bundle } p : E \longrightarrow M \\ \text{with a frame form } \theta = (\theta_{-1}, \theta_{-2}) \\ \text{of length one} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} M^{2n+1} \text{ with a complex} \\ \text{subbundle } T^{-1}M \subset TM,\ rank_{\mathbb{C}}T^{-1}M = n \\ \text{and a totally real bilinear pairing} \\ \{\cdot,\cdot\} : T^{-1}M \times T^{-1}M \longrightarrow TM/T^{-1}M, \\ \text{nondegenerate in every point} \end{array} \right\}.$$

Furthermore θ satisfies the structure equation if and only if the pairing $\{\cdot,\cdot\}$ is given by the Levi form, $\{\cdot,\cdot\} = G(\cdot, J\cdot)$.

Proof: It remains to prove that the pairing being given by the Levi form is equivalent to the frame form satisfying the structure equation.
Recall that for vector fields $\xi, \eta \in \Gamma(T^{-1}M) = \Gamma(H)$ the real Levi form is defined as

$$G(\xi, \eta) := -[\xi, J\eta] + T^{-1}M \in \Gamma(TM/_{T^{-1}M}) \text{ (see Subsection 2.1.2).}$$

Let ξ, η be two arbitrary vectors in $T_x^{-1}M$ and choose smooth sections $\widetilde{\xi}, \widetilde{\eta} \in \Gamma(T^{-1}E)$ with $dp_\varphi(\widetilde{\xi}) = \xi$ and $dp_\varphi(\widetilde{\eta}) = \eta$. Since the kernel of θ_{-2} is $T^{-1}E$ we have for the structure function

$$d\theta_{-2}(\widetilde{\xi}_\varphi, \widetilde{\eta}_\varphi) + \big[\theta_{-1}(\widetilde{\xi}_\varphi), \theta_{-1}(\widetilde{\eta}_\varphi)\big]_{\mathfrak{g}} = -\theta_{-2}\big([\widetilde{\xi}, \widetilde{\eta}](\varphi)\big) + \big[\theta_{-1}(\widetilde{\xi}_\varphi), \theta_{-1}(\widetilde{\eta}_\varphi)\big]_{\mathfrak{g}}.$$

Thus applying the isomorphism $\mathfrak{g}_{-2} \simeq T_xM/_{T_x^{-1}M}$ implied by θ_{-2} the vanishing of the structure function is equivalent to

$$\begin{aligned} dp \circ \theta_{-2}^{-1} \circ \theta_{-2}\big([\widetilde{\xi}, \widetilde{\eta}](\varphi)\big) &= dp \circ \theta_{-2}^{-1}\Big(\big[\theta_{-1}(\widetilde{\xi}_\varphi), \theta_{-1}(\widetilde{\eta}_\varphi)\big]_{\mathfrak{g}}\Big) \\ &= dp \circ \theta_{-2}^{-1}\Big(\big[\theta_{-1} \circ dp^{-1}(\xi), \theta_{-1} \circ dp^{-1}(\eta)\big]_{\mathfrak{g}}\Big) \\ &= \{\xi, \eta\}. \end{aligned}$$

However with the help of the real Levi form we have

$$\begin{aligned} dp \circ \theta_{-2}^{-1} \circ \theta_{-2}\big([\widetilde{\xi}, \widetilde{\eta}](\varphi)\big) &= dp\big([\widetilde{\xi}, \widetilde{\eta}](\varphi) + T_\varphi^{-1}E\big) \\ &= \big[dp(\widetilde{\xi}), dp(\widetilde{\eta})\big](x) + T_x^{-1}M \\ &= G_x\big(dp(\widetilde{\xi}), J \circ dp(\widetilde{\eta})\big) \\ &= G_x(\xi, J\eta). \end{aligned}$$

Hence the vanishing of the structure function is equivalent to the bilinear pairing being given by the Levi form, $\{\cdot, \cdot\} = G(\cdot, J\cdot)$.

□

Remark 4.1 *Since the torsion of a frame form of length one is homogeneous of degree zero it is automatically harmonic. Hence the prolongation procedure can be carried out.*

Remark 4.2 *Please note that so far we have not used the vanishing of the Nijenhuis tensor. Thus the prolongation procedure can also be used for partially integrable almost CR manifolds.*

4.4 The Prolongation

We now want to apply the prolongation procedure to CR manifolds. This section is again based on [CS00].
Let (M^{2n+1}, H, J, θ) be a strictly pseudo-convex CR manifold of dimension $2n + 1$. I.e. we have (see Subsections 2.1.1 and 2.1.2)

1. $H \subset TM$ is a real subbundle of codimension one.
2. $J : H \longrightarrow H$ is an almost complex bundle endomorphism, $J^2 = -1$.
3. The following integrability conditions hold:
   (a) $[JX, Y] + [X, JY] \in \Gamma(H)$ for all sections $X, Y \in \Gamma(H)$.

(b) The Nijenhuis tensor vanishes,

$$N_J(X,Y) := J\big([JX,Y]+[X,JY]\big) - [JX,JY] + [X,Y] = 0 \text{ for all } X,Y \in H.$$

4. $\theta \in \Omega^1(M^{2n+1},\mathbb{R})$ is a pseudo-hermitian form on (M,H,J) (that is $\theta_x \neq 0$ for all $x \in M$ and $\theta|_H = 0$).

5. The Levi form
$$G_\theta(\cdot,\cdot) := d\theta(\cdot, J\cdot) = \theta \circ G(\cdot,\cdot) : H \times H \longrightarrow \mathbb{R}$$
with $G(\cdot,\cdot) := -[\cdot, J\cdot] + H : H \times H \longrightarrow TM/_H$ is positive definite.

As we have seen in Subsection 2.1.2 the Levi form is also totally real. Thus by setting $T^{-1}M := H$ and

$$\{\cdot,\cdot\} := G(\cdot, J\cdot) : T^{-1}M \times T^{-1}M \longrightarrow TM/_{T^{-1}M}$$

we obtain a totally real bilinear pairing which is nondegenerate in every point.
Hence by Proposition 4.3 we obtain a G_0-principal bundle $p^1 : E^1 \longrightarrow M$ endowed with a frame form θ^1 of lenth one, which satisfies the structure equations and is harmonic by setting

$$E^1 := \left\{ (\varphi_1,\varphi_2)_x \;\middle|\; \begin{array}{c} x \in M,\ \varphi_1 : \mathfrak{g}_{-1} \longrightarrow T_x^{-1}M \text{ complex-linear isomorphism,} \\ \varphi_2 : \mathfrak{g}_{-2} \longrightarrow T_xM/T_x^{-1}M \text{ linear isomorphism with} \\ \{\varphi_1(X),\varphi_1(Y)\} = \varphi_2[X,Y]_{\mathfrak{g}} \text{ for all } X,Y \in \mathfrak{g}_{-1} \end{array} \right\}$$

and

$$\begin{array}{rcll} \theta^1_{-2}(\varphi)(\xi) & := & \varphi_2^{-1}([dp^1\xi]) & \text{for all } \xi \in T_\varphi E^1 = T_\varphi^{-2}E^1, \\ \theta^1_{-1}(\varphi)(\xi) & := & \varphi_1^{-1}(\underbrace{dp^1\xi}_{\in T_x^{-1}M}) & \text{for all } \xi \in d(p^1)^{-1}(T_x^{-1}M) = T_\varphi^{-1}E^1. \end{array}$$

I.e. $(E^1,p^1,M;\theta^1)$ is a harmonic P-frame bundle of degree one.
We can apply the prolongation procedure four times and obtain the regular Cartan bundle $(E^5,\pi^5,M;P)$, a P-principal bundle with the frame form θ^5 of length 5 and θ^5_{-2} is a Cartan connection with ∂^*-closed curvature, $\partial^*\Omega^\omega \equiv 0$, and $(\Omega^\omega)^l \equiv 0$ for $l \leq 0$.

4.5 Recovering the CR-Structure from the Cartan Bundle

Assume now we have given a regular Cartan geometry $(\mathcal{G},\pi,M;\omega)$ over a CR manifold M. We want to recover the CR-structure of M.
As we have started of constructing the Cartan bundle of a strictly pseudo-convex CR manifold (M,H,J,θ) by setting $T^{-1}M := H = Re(T_{10} \oplus \overline{T_{10}})$ we obviously recover the codimension one subbundle H from a given Cartan geometry $(\mathcal{G},\pi,M;\omega)$ over a CR manifold M via
$$H = T^{-1}M = d\pi\big(T^{-1}\mathcal{G}\big) = d\pi \circ \omega^{-1}(\mathfrak{g}^{-1}) = d\pi \circ \omega^{-1}(\mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_2).$$
And consequently
$$\mathbb{R}T \simeq TM/_{T^{-1}M} = T^{-2}M/_{T^{-1}M}.$$

According to the construction of the Cartan bundle the isomorphism $H \simeq \mathfrak{g}^{-1}/_{\mathfrak{p}}$ is complex linear and thus the almost complex bundle isomorphism $J : H \longrightarrow H$ can be recovered from

the complex structure of $\mathfrak{g}_{-1}$, $JX := d\pi \circ \omega_u^{-1}\big(i \cdot (\omega_u(\overline{X}) + \mathfrak{p})\big)$ where $\overline{X} \in T_u\mathcal{G}$ is a lift of the vector $X \in T_xM$. Please note that this definition of the endomorphism J is independent of the choice of $u \in \mathcal{G}_x$ since the Adjoint action of P on $\mathfrak{g}_{-1}$ preserves the complex structure as can be seen in Subsection 4.2.2.
The Levi form, defined by $G(X,Y) := -[X, JY] + T^{-1}M$, for $X, Y \in \Gamma(T^{-1}M)$, is totally real if and only if the first integrability condition, $[X,Y] - [JX, JY] \in \Gamma(T^{-1}M)$, holds for all sections $X, Y \in \Gamma(T^{-1}M)$, since

$$\begin{aligned} G(JX,Y) - G(J^2X, JY) &= -[JX, JY] + [J^2X, J^2Y] + T^{-1}M \\ &= [X,Y] - [JX, JY] + T^{-1}M. \end{aligned}$$

Now we want to study under which conditions the Nijenhuis tensor vanishes on H. So let ξ, η be local sections of $T^{-1}M$ with lifts $\overline{\xi}, \overline{\eta} \in \Gamma(T^{-1}\mathcal{G})$. We use the following notations:

$$\begin{aligned} &X(u) := \omega_{-1}(\overline{\xi}(u)),\ \ A(u) := \omega_{\mathfrak{p}}(\overline{\xi}(u)),\ \ A_0 = pr_{\mathfrak{g}_0}(A) \text{ and} \\ &Y(u) := \omega_{-1}(\overline{\eta}(u)),\ \ B(u) := \omega_{\mathfrak{p}}(\overline{\eta}(u)),\quad\ \ B_0 = pr_{\mathfrak{g}_0}(B). \end{aligned}$$

Thus we have $\overline{\xi}_u = \omega_u^{-1}(X) + \widetilde{A}_u$ and $\overline{\eta}_u = \omega_u^{-1}(Y) + \widetilde{B}_u$ and we can write

$$\begin{aligned} \omega_{-1}\big([\overline{\xi}, \overline{\eta}]\big)_u &= -d\omega_{-1}(\overline{\xi}, \overline{\eta})_u + \overline{\xi}\big(\omega_{-1}(\eta)\big)_u - \overline{\eta}\big(\omega_{-1}(\xi)\big)_u \\ &= -d\omega_{-1}(\overline{\xi}, \overline{\eta})_u + \overline{\xi}\big(Y(u)\big) - \overline{\eta}\big(X(u)\big) \\ &= -\,\big(\Omega^{\omega}_{-1}(\overline{\xi}, \overline{\eta})_u - pr_{\mathfrak{g}_{-1}}\big([\omega(\overline{\xi}), \omega(\overline{\eta})]_{\mathfrak{g}}\big)\big) + \overline{\xi}\big(Y(u)\big) - \overline{\eta}\big(X(u)\big) \\ &= -\Omega^{\omega}_{-1}(\overline{\xi}, \overline{\eta})_u + [\omega_{-1}(\overline{\xi}), \omega_0(\overline{\eta})]_{\mathfrak{g}} + [\omega_0(\overline{\xi}), \omega_{-1}(\overline{\eta})]_{\mathfrak{g}} + \overline{\xi}\big(Y(u)\big) - \overline{\eta}\big(X(u)\big) \\ &= -\Omega^{\omega}_{-1}(\overline{\xi}, \overline{\eta})_u + [X(u), B_0(u)]_{\mathfrak{g}} + [A_0(u), Y(u)]_{\mathfrak{g}} + \overline{\xi}\big(Y(u)\big) - \overline{\eta}\big(X(u)\big). \end{aligned}$$

Since the curvature is horizontal this is

$$\begin{aligned} \omega_{-1}\big([\overline{\xi}, \overline{\eta}]\big)_u = {} & -\Omega^{\omega}_{-1}\big(\omega^{-1}(X(u)), \omega^{-1}(Y(u))\big)_u + [X(u), B_0(u)]_{\mathfrak{g}} + [A_0(u), Y(u)]_{\mathfrak{g}} \\ & + \overline{\xi}\big(Y(u)\big) - \overline{\eta}\big(X(u)\big). \end{aligned}$$

The Nijenhuis tensor vanishes if and only if $\omega_{-1}\left([\overline{\xi}, \overline{\eta}] - [\overline{J\xi}, \overline{J\eta}] + \overline{J\big([J\xi, \eta] + [\xi, J\eta]\big)}\right)$ vanishes. Using the equation above and keeping in mind that $d\pi\big(\omega^{-1}(iX(u)) + \widetilde{A}_u\big) = J\xi$ this can be written as

$$\begin{aligned} & \omega_{-1}\left([\overline{\xi}, \overline{\eta}] - [\overline{J\xi}, \overline{J\eta}] + \overline{J\big([J\xi, \eta] + [\xi, J\eta]\big)}\right) \\ = {} & -\Omega^{\omega}_{-1}\big(\omega^{-1}(X), \omega^{-1}(Y)\big)_u + \underline{[X(u), B_0(u)]_{\mathfrak{g}}} + \underbrace{[A_0(u), Y(u)]_{\mathfrak{g}} + \overline{\xi}\big(Y(u)\big)} \underline{-\overline{\eta}\big(X(u)\big)} \\ & + \Omega^{\omega}_{-1}\big(\omega^{-1}(iX), \omega^{-1}(iY)\big)_u \underline{-[iX(u), B_0(u)]_{\mathfrak{g}}} \overbrace{-[A_0(u), iY(u)]_{\mathfrak{g}} - \overline{J\xi}\big(iY(u)\big)} + \overbrace{\overline{J\eta}\big(iX(u)\big)} \\ & + i\Big(- \Omega^{\omega}_{-1}\big(\omega^{-1}(iX), \omega^{-1}(Y)\big)_u + \underline{[iX(u), B_0(u)]_{\mathfrak{g}}} + \overbrace{[A_0(u), Y(u)]_{\mathfrak{g}} + \overline{J\xi}\big(Y(u)\big)} \underline{-\overline{\eta}\big(iX(u)\big)}\Big) \\ & + i\Big(- \Omega^{\omega}_{-1}\big(\omega^{-1}(X), \omega^{-1}(iY)\big)_u + \underline{[X(u), B_0(u)]_{\mathfrak{g}}} + \underbrace{[A_0(u), iY(u)]_{\mathfrak{g}} + \overline{\xi}\big(iY(u)\big)} \overbrace{-\overline{J\eta}\big(X(u)\big)}\Big). \end{aligned}$$

All terms except for the following cancel out.

$$\begin{aligned} \omega_{-1}\left([\overline{\xi}, \overline{\eta}] - [\overline{J\xi}, \overline{J\eta}] + \overline{J\big([J\xi, \eta] + [\xi, J\eta]\big)}\right) = {} & -\Omega^{\omega}_{-1}\big(\omega^{-1}(X), \omega^{-1}(Y)\big)_u \\ & + \Omega^{\omega}_{-1}\big(\omega^{-1}(iX), \omega^{-1}(iY)\big)_u \\ & - i\Omega^{\omega}_{-1}\big(\omega^{-1}(iX), \omega^{-1}(Y)\big)_u \\ & - i\Omega^{\omega}_{-1}\big(\omega^{-1}(X), \omega^{-1}(iY)\big)_u \end{aligned}$$

However since both $\omega^{-1}(X)$ and $\omega^{-1}(Y)$ are of degree -1 the $\mathfrak{g}_{-1}$-component of the curvature is exactly the homogeneous component of degree one,

$$\Omega^{\omega}_{-1}\big(\omega^{-1}(X),\omega^{-1}(Y)\big)_u = (\Omega^{\omega})^1\big(\omega^{-1}(X),\omega^{-1}(Y)\big)_u.$$

$$\begin{aligned}\omega_{-1}\Big([\overline{\xi},\overline{\eta}]-[\overline{J\xi},\overline{J\eta}]+\overline{J\big([J\xi,\eta]+[\xi,J\eta]\big)}\Big) \;=\; & -(\Omega^{\omega})^1\big(\omega^{-1}(X),\omega^{-1}(Y)\big)_u \\ & +(\Omega^{\omega})^1\big(\omega^{-1}(iX),\omega^{-1}(iY)\big)_u \\ & -i(\Omega^{\omega})^1\big(\omega^{-1}(iX),\omega^{-1}(Y)\big)_u \\ & -i(\Omega^{\omega})^1\big(\omega^{-1}(X),\omega^{-1}(iY)\big)_u\end{aligned}$$

Now extending the map $\kappa^1_u := (\Omega^{\omega})^1\big(\omega^{-1}(\cdot),\omega^{-1}(\cdot)\big)_u : \bigwedge^2 \mathfrak{g}_- \longrightarrow \mathfrak{g}$ to the complexification $\mathfrak{g}^{\mathbb{C}}$ this is still ∂- and ∂^*-closed and therefore the harmonic representative for a cohmology class in $H^2_1(\mathfrak{g}^{\mathbb{C}}_-,\mathfrak{g}^{\mathbb{C}})$. According to [Kost61] these representatives do not preserve the two irreducible submodules into which the $\mathfrak{g}_0$-module $\mathfrak{g}^{\mathbb{C}}_{-1}$ splits. However if the Nijenhuis tensor vanishes, κ^1_u would preserve these submodules, and thus κ^1_u would have to vanish. In a similar way (see [CS00] Section 4.16) dealing with κ^2 leads to the vanishing of the Nijenhuis tensor being equivalent to the Cartan connection being torsion free, $t = \Omega^{\omega}_- = 0$.

Chapter 5

The Fefferman Space According to [CG08]

Although we already explained the construction of Fefferman spaces according to [BL04] in chapter 2 we now want to discuss another construction from [CG08]. This second approach uses widely the Cartan geometry and leads to a strong relationship between the Cartan geometries of a CR manifold M and the corresponding Fefferman space $\mathcal{F} := \mathcal{G}/_{\widetilde{P}\cap G}$, namely

$$\widetilde{\mathcal{G}} = \mathcal{G} \times_{G\cap\widetilde{P}} \widetilde{P},$$

where the tilde denotes the objects of the Fefferman space. The Cartan connection of the Fefferman space will be given as

$$\widetilde{\omega}_{[u,\widetilde{p}]} = Ad(\widetilde{p}^{-1}) \circ \pi^*_{\mathcal{G}}\omega + \pi^*_{\widetilde{P}}\omega_{\widetilde{P}}.$$

However we want to point out, that in order to achieve this a rather strong assumption, the existence of an $(n+2)$nd root of the anticanonical complex line bundle, has to be made. This root exists for CR manifolds embedded in $\mathbb{C}^{n+1}$ globally and locally we have this root for any CR manifold. However, since we are interested in boundaries we need a global construction of the Fefferman space. So for boundary considerations it is very helpful to have both constructions at hand.
This chapter is based on [CG08].
As before we use the Lie group

$$\begin{aligned} G &:= PSU(1,n+1) \\ &= SU(1,n+1)/_{\{x\cdot Id \mid x\in\mathbb{C}, x^{2+n}=1\}} \\ &= SU(1,n+1)/_{\mathbb{Z}_{n+2}}, \end{aligned}$$

although most of the time we will write instead $G = SU(1,n+1)$ in order to simplify calculations and notations, keeping in mind that we actually work modulo $\mathbb{Z}_{n+2}$. Its Lie algebra is 2-graded, $LA(G) = \mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{p}$, with $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+ = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ being the Lie algebra of the subgroup $P \subset G$ given by

$$P = \{p \in G \mid Ad(p)(\mathfrak{g}^j) \subset \mathfrak{g}^j \text{ for all } j = -2,\dots,2\}\ .$$

Remark 5.1 *This construction can actually be done for arbitrary signature $(p+1,q+1)$. However since we only need the case of Lorentzian signature we will restrict to this case to make notations easier to handle. Sometimes we will give more general calculations using the arbitrary signature $(p+1,q+1)$ just for reminding that there is no need of restriction.*

5.1 The Homogeneous Model for CR Manifolds

With G and P as above the homogeneous space $G/_P$ is endowed with a CR structure as we will see now.

1. $\mathfrak{g}_{-1} + \mathfrak{p}$ is a nondegenerate subspace of $\mathfrak{g}/_{\mathfrak{p}}$ of codimension one. The Levi form is given by:

   $$\begin{aligned} \mathcal{L} : (\mathfrak{g}_{-1}+\mathfrak{p}) \times (\mathfrak{g}_{-1}+\mathfrak{p}) &\longrightarrow (\mathfrak{g}/_{\mathfrak{p}})/_{(\mathfrak{g}_{-1}+\mathfrak{p})} = \mathfrak{g}/_{\mathfrak{g}^{-1}} \\ X, Y &\mapsto \mathcal{L}(X,Y) := [X,Y]_{\mathfrak{g}} + \underbrace{(\mathfrak{g}_{-1}+\mathfrak{p})}_{=\mathfrak{g}^{-1}} = [X,Y]_{\mathfrak{g}} + \mathfrak{g}^{-1}. \end{aligned}$$

   We have especially $\mathcal{L}(E_{-1}(X)+\mathfrak{p}, E_{-1}(Y)+\mathfrak{p}) = E_{-2}\big(2Im(Y^*\cdot X)\big) + \mathfrak{g}^{-1}$. I.e. the Levi form is nondegenerate.

2. Since $\mathfrak{g}^{-1} = \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_2$ is invariant under the Adjoint action of P and is of codimension one, we can prolong $\mathfrak{g}^{-1}$ to a codimension-1 subbundle $H \subset T\left(G/_P\right)$ on which the Levi form is nondegenerate. ([Ham07] Theorem 6.4.2.)

3. We still need a P-invariant complex structure J on H respectively on $\mathfrak{g}^{-1}$. We set

$$J(E_{-1}(X) + \mathfrak{p}) := E_{-1}(iX) + \mathfrak{p}.$$

   The integrability conditions have to hold:

   - $[JX, Y]_\mathfrak{g} + [X, JY]_\mathfrak{g} \overset{!}{\in} \mathfrak{g}^{-1}$ for all $X, Y \in \mathfrak{g}^{-1}$.

     This is true because the $\mathfrak{g}_{-2}$-component of the term $[JX, Y]_\mathfrak{g} + [X, JY]_\mathfrak{g}$ for $X = E_{-1}(X) + \mathfrak{p}$ and $Y = E_{-1}(Y) + \mathfrak{p}$ is given by :

$$\begin{aligned} & [JE_{-1}(X), E_{-1}(Y)]_\mathfrak{g} + [E_{-1}(X), JE_{-1}(Y)]_\mathfrak{g} \\ = \; & E_{-2}\big(2Im(Y^* \cdot iX)\big) + E_{-2}\big(2Im((iY)^* \cdot X)\big) \\ = \; & E_{-2}\big(2Im(i(Y^* \cdot X - Y^* \cdot X))\big) \\ = \; & 0. \end{aligned}$$

     Hence we have $[JX, Y]_\mathfrak{g} + [X, JY]_\mathfrak{g} \in \mathfrak{g}^{-1}$ for all $X, Y \in \mathfrak{g}^{-1}$.

   - The Nijenhuis tensor has to vanish on $\mathfrak{g}^{-1}$. We choose arbitrary representatives $E_{-1}(X) + E_0(a, A) + \mathfrak{p}_+$ and $E_{-1}(Y) + E_0(b, B) + \mathfrak{p}_+$ for $X, Y \in \mathfrak{g}^{-1}$ and calculate the Nijenhuis tensor.

$$\begin{aligned} N_J(X,Y) \; = \; & J\Big(\big[JX, Y\big]_\mathfrak{g} + \big[X, JY\big]_\mathfrak{g}\Big) - \big[JX, JY\big]_\mathfrak{g} + \big[X, Y\big]_\mathfrak{g} \\ = \; & J\big(\big[E_{-1}(iX) + E_0(a, A), E_{-1}(Y) + E_0(b, B)\big]_\mathfrak{g} \\ & + \big[E_{-1}(X) + E_0(a, A), E_{-1}(iY) + E_0(b, B)\big]_\mathfrak{g}\big) \\ & - \big[E_{-1}(iX) + E_0(a, A), E_{-1}(iY) + E_0(b, B)\big]_\mathfrak{g} \\ & + \big[E_{-1}(X) + E_0(a, A), E_{-1}(Y) + E_0(b, B)\big]_\mathfrak{g} + \mathfrak{p} \\ = \; & J\big(\underbrace{\big[E_{-1}(iX), E_{-1}(Y)\big]_\mathfrak{g} + \big[E_{-1}(X), E_{-1}(iY)\big]_\mathfrak{g}}_{=0} \\ & + \big[E_{-1}(X + iX), E_0(b, B)\big]_\mathfrak{g} + \big[E_0(a, A), E_{-1}(Y + iY)\big]_\mathfrak{g}\big) \\ & \underline{-\big[E_{-1}(iX), E_{-1}(iY)\big]_\mathfrak{g}} - \big[E_{-1}(iX), E_0(b, B)\big]_\mathfrak{g} \\ & -\big[E_0(a, A), E_{-1}(iY)\big]_\mathfrak{g} + \underline{\big[E_{-1}(X), E_{-1}(Y)\big]_\mathfrak{g}} \\ & + \big[E_{-1}(X), E_0(b, B)\big]_\mathfrak{g} + \big[E_0(a, A), E_{-1}(Y)\big]_\mathfrak{g} + \mathfrak{p} \end{aligned}$$

     The underlined terms cancel out and we obtain

$$\begin{aligned} N_J(X,Y) \; = \; & J\Big(\big[E_{-1}(X + iX), E_0(b, B)\big]_\mathfrak{g} + \big[E_0(a, A), E_{-1}(Y + iY)\big]_\mathfrak{g}\Big) \\ & + \big[E_{-1}(X - iX), E_0(b, B)\big]_\mathfrak{g} + \big[E_0(a, A), E_{-1}(Y - iY)\big]_\mathfrak{g} + \mathfrak{p} \\ = \; & \big[E_{-1}(iX - X), E_0(b, B)\big]_\mathfrak{g} + \big[E_0(a, A), E_{-1}(iY - Y)\big]_\mathfrak{g} \\ & + \big[E_{-1}(X - iX), E_0(b, B)\big]_\mathfrak{g} + \big[E_0(a, A), E_{-1}(Y - iY)\big]_\mathfrak{g} + \mathfrak{p} \\ = \; & 0 + \mathfrak{p}. \end{aligned}$$

     So the Nijenhuis tensor vanishes as wanted.

Consequently we have a CR structure on $G/_P$ of signature (p,q). On the P-principal bundle $G \longrightarrow G/_P$ the Maurer Cartan form $\omega_G \in \Omega^1(G,\mathfrak{g})$ defined as left translation to the neutral element as base point, $\omega_G(X(a)) := dL_{a^{-1}}(X(a)) \in \mathfrak{g}$, is a Cartan connection.

Another possibility to describe the homogeneous model is given by using the light cone without the origin in $\mathbb{C}^{p+1,q+1}$,

$$\mathcal{C} := \left\{ v \in \mathbb{C}^{p+1,q+1} \mid v \neq 0, \langle v, v\rangle_{p+1,q+1} = 0 \right\}.$$

We denote the complex projectivization by

$$p : \mathbb{C}^{p+1,q+1} \longrightarrow \mathbb{C}^{p+1,q+1}/_{\mathbb{C}^*} = \mathbb{C}P^{p+q+1}$$

and obtain $M := p(\mathcal{C}) = \mathcal{C}/_{\mathbb{C}^*}$ as the set of all complex null lines $\ell \subset \mathbb{C}^{p+1,q+1}$. M is a smooth hypersurface in $\mathbb{C}P^{p+q+1}$.

The CR structure in the point $\ell \in M$, where ℓ is a complex null line in $\mathbb{C}^{p+1,q+1}$, is defined as $H_\ell M := d\left(p|_{\mathcal{C}}\right)\left(\ell^\perp\right)$ and is therefore isomorphic to $\ell^\perp/_\ell$.

Up to a non zero multiple the Levi form at the point ℓ corresponds to the Hermitian product on $\ell^\perp/_\ell$ induced by $\langle\cdot,\cdot\rangle_{p+1,q+1}$. Consequently the Levi form is nondegenerate and of signature (p,q).

Lets take a look at the G-action on M. G denotes as above the special unitary group of the complex vector space $\mathbb{C}^{p+1,q+1}$.

$$\begin{aligned} G &= SU(p+1,q+1) \\ &= \left\{ A \in Gl(p+q+2,\mathbb{C}) \;\middle|\; \begin{array}{l} \langle Ax, Ay\rangle_{p+1,q+1} = \langle x, y\rangle_{p+1,q+1} \text{ for all } x, y \in \mathbb{C}^{p+1,q+1}, \\ det(A) = 1 \end{array} \right\} \end{aligned}$$

The G-action on $\mathbb{C}^{p+1,q+1}$ restricts to the light cone, since the action preserves the Hermitian product. Furthermore the lines of the light cone are again mapped to lines of the cone and the G-action is transitive on $\mathcal{C}$. Hence the action descends to a smooth transitive left action of G on M. In addition G acts on M by CR automorphisms since the Hermitian form $\langle\cdot,\cdot\rangle_{p+1,q+1}$ is preserved and the CR structure on M is completely defined by $\langle\cdot,\cdot\rangle_{p+1,q+1}$.

Denoting with P the stabilizer of a fixed null line $\ell \subset \mathbb{C}^{p+1,q+1}$, we can identify M with the homogeneous space $G/_P$.

However the centre of the G-action on M is not trivial. For the elements of the centre and for all points x of all null lines we can find an $\alpha \in \mathbb{C}$ such that $A \cdot x = \alpha x$. This is of cause true for A being a multiple of the identity. And there are no further solutions, which can be seen by choosing for x vectors of the shape $e_i + e_j$ with $i \in \{1,\ldots,p+1\}$ and $j \in \{p+2,\ldots p+q\}$. Hence A has to be a multiple of the identity $A = \alpha \cdot Id$. And since the determinant of A has to be one, $det(A) = \alpha^{n+2} \overset{!}{=} 1$, we obtain for the centre

$$Z(G) = \left\{ e^{\frac{2\pi i k}{n+2}} Id \;\middle|\; k = 0, 1, \ldots, n+1 \right\} \simeq \mathbb{Z}_{n+2}.$$

We set $\underline{G} := G/_{Z(G)}$ and $\underline{P} := P/_{Z(G)}$. Hence we can write M as a homogeneous space of its group of CR automorphisms, $M = \underline{G}/_{\underline{P}}$.

5.2 The Fefferman Space of the Homogeneous Model

The corresponding Fefferman space of the homogeneous model $M = \underline{G}/_{\underline{P}} = \mathcal{C}/_{\mathbb{C}^*}$ results from the real version. Seen as a real vector space, $\mathbb{C}^{p+1,q+1}$ is equipped with the inner product $\langle\cdot,\cdot\rangle_{\mathbb{R}} = Re\langle\cdot,\cdot\rangle_{p+1,q+1}$ induced by the original hermitian product of signature $(p+1, q+1)$.
Real projectivization gives $P_{\mathbb{R}}\mathbb{C}^{p+1,q+1} = \mathbb{C}^{p+1,q+1}/_{\sim}$ with $v \sim w$ be given if and only if there exists a $\lambda \in \mathbb{R}$ such that $v = \lambda w$ is true. $\widetilde{p} : \mathbb{C}^{p+1,q+1} \longrightarrow P_{\mathbb{R}}\mathbb{C}^{p+1,q+1}$ denotes the projection. We define

$$\widetilde{M} := \widetilde{p}(\mathcal{C}).$$

So $\widetilde{M}$ is the space of all real null lines with respect to $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ and this is a smooth hypersurface in $\mathbb{R}P^{2p+2q+3}$. $\mathcal{C} \longrightarrow \widetilde{M}$ is a principal bundle with fibre group $\mathbb{R}^*$. Since any real null line is contained in the complex null line generated by it, we obtain the smooth projection $\widetilde{M} \longrightarrow M$. This is a fibre bundle over M with the fibre $\mathbb{R}P^1 \simeq S^1$ the space of all real lines in $\mathbb{C}$.
Let $v \in \mathcal{C}$ be fix, $\widetilde{\ell} := \mathbb{R}v$ denotes the real line defined by v. $\widetilde{\ell}^{\perp}$ denotes the orthogonal complement of $\widetilde{\ell}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathbb{R}}$. Then $T_{\widetilde{l}}\widetilde{M}$ can be written as $d\big(\widetilde{p}\big|_{\mathcal{C}}\big)_v\left(\widetilde{l}^{\perp}\right) = \left(\widetilde{l}^{\perp}/_{\widetilde{l}}\right)$. The inner product on $\mathcal{C}$ induces a product on the tangent space $T_{\widetilde{\ell}}\widetilde{M}$ as follows. For $\widetilde{X}, \widetilde{Y} \in T_{\widetilde{\ell}}\widetilde{M}$ we define $\langle\widetilde{X}, \widetilde{Y}\rangle := \langle X, Y\rangle_{\mathbb{R}}$ with $X, Y \in T_v\mathcal{C}$ satisfying $d\widetilde{p}_v(X) = \widetilde{X}$ and $d\widetilde{p}_v(Y) = \widetilde{Y}$. This is well defined since $\langle\ell,\ell\rangle_{\mathbb{R}} = 0$.
By choosing another element $v' \in \mathcal{C}$ we obtain a conformally changed product. Consequently the conformal class $[\langle\cdot,\cdot\rangle]$ is independent of the choice of $v \in \mathcal{C}$.
Please note that this is exactly the construction of the Möbius space, the homogeneous model of conformal geometry (see for example [Feh05]).
Let $\widetilde{G}$ be the connected component of the identity of the orthogonal group which we denote by $O_c\big(\mathbb{C}^{p+1,q+1}, \langle\cdot,\cdot\rangle_{\mathbb{R}}\big)$. Analogously to the complex case $\widetilde{P} \subset \widetilde{G}$ denotes the stabilizer of a real null line. We obtain a transitive action of $\widetilde{G}$ on $\widetilde{M}$, which is given by conformal isomorphisms. Hence $\widetilde{M}$ can be identified with the homogeneous space, $\widetilde{M} = \widetilde{G}/_{\widetilde{P}}$.
Since $G \subset \widetilde{G}$ acts transitively on the lightcone without zero, G acts transitively on $\widetilde{M}$ as well. Let $\widetilde{\ell}$ be a real null line in $\mathcal{C}$ and ℓ be the complex null line generated by it. As above $\widetilde{P} = stab_{\widetilde{G}}(\widetilde{\ell})$ is the stabilizer of $\widetilde{\ell}$ and $P = stab_G(\ell)$ is the stabiliser of $\ell = \mathbb{C}\cdot\widetilde{\ell}$. Obviously $\widetilde{P}$ preserves $\ell = \mathbb{C}\cdot\widetilde{\ell}$. Hence $G \cap \widetilde{P} \subset P$. And $G \cap \widetilde{P} = stab_G(\widetilde{\ell})$ is the stabiliser of a real null line. Consequently we can write as well

$$\widetilde{M} = \widetilde{G}/_{\widetilde{P}} \simeq G/_{(G\cap\widetilde{P})}.$$

5.3 The Homogeneous Model and the Canonical Line Bundle

Since in [Bau99] and [BL04] the Fefferman space is constructed with the help of the canonical line bundle, the relationship between the canonical line bundle and the light cone shall be studied now. This will result in a motivation for the construction according to [CG08].
Recall that the canonical line bundle is obtained by taking the annihilator of $\overline{T_{10}}$ in the complexified cotangent bundle and forming its $(n+1)$st exterior power.

$$\mathcal{K} := \left\{\omega \in \bigwedge\nolimits^{n+1}(TM^{\mathbb{C}})^* \;\middle|\; i_V\omega = 0 \text{ for all } V \in \overline{T_{10}}\right\}.$$

Using the action $\widetilde{Ad}^* = Ad^{\mathbb{C}}_{\mathfrak{g}_{-2}} \wedge \left(Ad^{\mathbb{C}}_{\mathfrak{g}_{-1}}\right)^{\wedge n}$ of P on $\mathfrak{g}^{\mathbb{C}}_{-2} \wedge \mathfrak{g}^{\wedge n}_{10}$ as in Subsection 4.2.2 we have the following isomorphism for the dual of the canonical line bundle of our homogeneous CR manifold $M = G/_P = \mathcal{C}/_{\mathbb{C}^*}$.

$$\begin{array}{rcl} G \times_{\left[P,\widetilde{Ad}^*\right]} \mathfrak{g}^{\mathbb{C}}_{-2} \wedge \mathfrak{g}^{\wedge n}_{10} & \longrightarrow & \mathcal{K}^* \\ [u, X_{-2} \wedge X_{-11} \wedge \cdots \wedge X_{-1n}] & \mapsto & pr_{\mathbb{C}T} \circ d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})^{-1}_u (X_{-2}) \\ & & \wedge \left(d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})^{-1}_u (X_{-11})\right) \\ & & \wedge \cdots \wedge \left(d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})^{-1}_u (X_{-1n})\right) \end{array}$$

As we have seen, we have $\widetilde{Ad}^*(A) = a^{-n-2}$ for matrices $A = \begin{pmatrix} a & -aZ^*B & b \\ 0 & B & Z \\ 0 & 0 & \overline{a}^{-1} \end{pmatrix} \in P.$

Furthermore for a fixed point $v_0 \in \ell$ and any point v of the light cone, $v \in \mathcal{C}$, we have a transferring element $A_v \in G$ with $A_v \cdot v_0 = v$. Now for any complex number $z \in \mathbb{C}^*$ the point $A_v^{-1}(zv) = zv_0$ is an element of the complex line ℓ and hence we have an element $A_z \in P$ such that $R_{A_z} A_v = A_v A_z = A_{zv}$ is the transferring element from v_0 to zv. Since the first entry of the first row of A_z has to be z, any such matrix A_z fulfills $\widetilde{Ad}^*(A_z) = z^{-n-2}$ as we have seen in Subsection 4.2.2. So we obtain the next isomorphism for a fixed basis (η) of $\mathfrak{g}^{\mathbb{C}}_{-2} \wedge \mathfrak{g}^{\wedge n}_{10}$

$$\begin{array}{rcl} \mathcal{C} \times_{[\mathbb{C}^*, z^{-n-2}]} \mathbb{C}^* & \longrightarrow & G \times_{\left[P,\widetilde{Ad}^*\right]} \mathfrak{g}^{\mathbb{C}}_{-2} \wedge \mathfrak{g}^{\wedge n}_{10} \\ [v, \lambda] & \mapsto & [A_v, \lambda\eta]. \end{array}$$

This is well-defined since

$$\begin{array}{rcl} [v, \lambda] & = & [zv, z^{n+2}\lambda] \\ & \mapsto & [A_{zv}, z^{n+2}\lambda\eta] \\ & = & [R_{A_z} A_v, \widetilde{Ad}^*(A_z^{-1})\lambda\eta] \\ & = & [A_v, \lambda\eta] \end{array}$$

and for any matrix A of the center of G we have $\widetilde{Ad}^*(A) = 1$.
Hence we have

$$\mathcal{K}^* \simeq \mathcal{C} \times_{[\mathbb{C}^*, z^{-n-2}]} \mathbb{C}^*.$$

5.4 The Fefferman Space

Note that the Cartan geometry of some CR manifold (M, H, J) is a $\underbrace{P/_{Z(G)}}_{=\underline{P}}$-principal bundle $(\underline{\mathcal{G}}, \underline{\pi}, M; \underline{\omega})$ joined by the normal Cartan connection, which is $\underline{P}$-equivariant, reproduces the generators of the fundamental vector fields and defines a trivialization of $T\underline{\mathcal{G}}$. $\underline{\omega}$ is uniquely defined up to isomorphisms by the normalization conditions:

- $\underline{\omega}$ has ∂^*-closed curvature, that is $\partial^* \circ \Omega^{\underline{\omega}} \equiv 0$.
- The homogeneous components of the curvature of degree less or equal to zero vanish, $(\Omega^{\underline{\omega}})^l \equiv 0$ for $l \leq 0$.
- $\underline{\omega}$ is torsion free, that is to say $t := pr_{\mathfrak{g}_-} \circ \Omega^{\underline{\omega}} = 0$.

However later it will be useful to have P as structure group to our disposal. To enlarge the Cartan bundle $\underline{\mathcal{G}}$ to a P-principal bundle, we will use a $(n+2)$nd root of the canonical complex line bundle as we will explain in this section. This root generally exists locally.

Though globally its existence is a restriction entailing the existence of a conformal spin structure for the associated Fefferman space (details on this spin structure can be found in [CG08]).

Let us now assume that we have given a strictly pseudo-convex CR manifold (M^{2n+1}, H, θ) joined by a complex line bundle $\mathcal{E}(1,0) \longrightarrow M$ together with a duality between $\mathcal{E}(1,0)^{\otimes(n+2)}$ and the canonical complex line bundle of M

$$\mathcal{K} := \{\omega \in \bigwedge^{n+1}(TM^{\mathbb{C}})^* \mid i_V\omega = 0 \text{ for all } V \in \overline{T_{10}}\}.$$

Since the tangent bundle of M is an associated bundle of the Cartan bundle the dual of the canonical complex line bundle is also associated to the Cartan bundle. To see this in detail, we will write down several isomorphisms and the actions of $\underline{P}$ on the spaces needed.

$$\begin{array}{rcl} Iso_1 : \underline{\mathcal{G}} \times_{\underline{P}} \mathfrak{g}/_{\mathfrak{p}} & \longrightarrow & TM \\ {[u, X]} & \mapsto & d\underline{\pi} \circ \underline{\omega}_u^{-1}(X) \end{array}$$

Here $\underline{P}$ acts by the Adjoint action on $\mathfrak{g}/_{\mathfrak{p}}$ and we have $[u, X] = [R_p u, Ad(p^{-1})X]$. A slight modification gives the structure of the graded tangent space.

$$\begin{array}{rcl} Iso_2 : \underline{\mathcal{G}} \times_{[\underline{P},(Ad_{\mathfrak{g}_{-2}}, Ad_{\mathfrak{g}_{-1}})]} \big(\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}\big) & \longrightarrow & Gr(TM) = T^{-2}M/_{T^{-1}M} \oplus T^{-1}M \\ {[u, X_{-2} \oplus X_{-1}]} & \mapsto & \big[d\underline{\pi} \circ \underline{\omega}_u^{-1}(X_{-2})\big] \oplus d\underline{\pi} \circ \underline{\omega}_u^{-1}(X_{-1}) \end{array}$$

As we have seen in Section 4.5, the graded tangent space mirrors the CR structure of our manifold.

$$\begin{array}{rcl} TM = \mathbb{R}T \oplus H & \longrightarrow & T^{-2}M/_{T^{-1}M} \oplus T^{-1}M \\ \lambda T \oplus X & \mapsto & [\lambda T] \oplus X \end{array}$$

I.e. we can write

$$\begin{array}{rcl} Iso_3 : \underline{\mathcal{G}} \times_{[\underline{P},(Ad_{\mathfrak{g}_{-2}}, Ad_{\mathfrak{g}_{-1}})]} \big(\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}\big) & \longrightarrow & \mathbb{R}T \oplus H \\ {[u, X_{-2} \oplus X_{-1}]} & \mapsto & pr_{\mathbb{R}T} \circ d\underline{\pi} \circ \underline{\omega}_u^{-1}(X_{-2}) \oplus d\underline{\pi} \circ \underline{\omega}_u^{-1}(X_{-1}). \end{array}$$

Dualization gives the next isomorphism. Recall that the dual of the restriction of the adjoint action to $\mathfrak{g}_j$ is given by $Ad^*_{\mathfrak{g}_j} = (pr_{\mathfrak{g}_j} \circ Ad)^* = pr_{\mathfrak{g}_{-j}} \circ Ad = Ad_{\mathfrak{g}_{-j}}$ (see Subsection 4.2.2).

$$\begin{array}{rcl} Iso_3^* : \underline{\mathcal{G}} \times_{[\underline{P},(Ad_{\mathfrak{g}_2}, Ad_{\mathfrak{g}_1})]} \big(\mathfrak{g}_2 \oplus \mathfrak{g}_1\big) & \longrightarrow & Gr(TM)^* = \big(\mathbb{R}T\big)^* \oplus H^* \\ {[u, X_2 \oplus X_1]} & \mapsto & \big(pr_{\mathbb{R}T} \circ d\underline{\pi} \circ \underline{\omega}_u^{-1}(X_2^*)\big)^* \oplus \big(d\underline{\pi} \circ \underline{\omega}_u^{-1}(X_1^*)\big)^* \end{array}$$

We obtain the isomorphism to the canonical complex line bundle as the $(n+1)$st exterier power of the annihilator of the subbundle $\overline{T_{10}}$ using the notations known from Subsection 4.2.2.

$$\begin{array}{rcl} Iso_4 : \underline{\mathcal{G}} \times_{\left[\underline{P}, \widetilde{Ad} = Ad_{\mathfrak{g}_2^{\mathbb{C}}} \wedge \left(Ad_{\mathfrak{g}^*_{10}}\right)^{\wedge n}\right]} \mathfrak{g}_2^{\mathbb{C}} \wedge (\mathfrak{g}^*_{10})^{\wedge n} & \longrightarrow & \mathcal{K} \\ {[u, X_2 \wedge X^*_{11} \wedge \cdots \wedge X^*_{1n}]} & \mapsto & \big(pr_{\mathbb{C}T} \circ d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})_u^{-1}(X_2^*)\big)^* \\ & & \wedge \big(d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})_u^{-1}(X_{11})\big)^* \\ & & \wedge \cdots \wedge \big(d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})_u^{-1}(X_{1n})\big)^* \end{array}$$

Recall that we denote with $\mathfrak{g}_{10}$ the subspace $\mathfrak{g}_{10} := \{X - iJX \mid X \in \mathfrak{g}_{-1}\} \subset \mathfrak{g}^{\mathbb{C}}_{-1}$, where J denotes the complex structure on $\mathfrak{g}_{-1}$ and i the one coming from the complexification. Hence a basis of $\mathfrak{g}_{10}$ is given by

$$\big(E_{-1}(e_1) - iE_{-1}(Je_1), \ldots, E_{-1}(e_n) - iE_{-1}(Je_n)\big).$$

Now we consider the dual of the canonical complex line bundle.

$$
\begin{array}{rcl}
Iso_4^* : \underline{\mathcal{G}} \times_{\left[\underline{P},\widetilde{Ad}^*=Ad_{\mathfrak{g}^{\mathbb{C}}_{-2}}\wedge\left(Ad_{\mathfrak{g}_{10}}\right)^{\wedge n}\right]} \mathfrak{g}^{\mathbb{C}}_{-2} \wedge \mathfrak{g}_{10}^{\wedge n} & \longrightarrow & \mathcal{K}^* \\
{[u, X_{-2} \wedge X_{11} \wedge \cdots \wedge X_{1n}]} & \mapsto & pr_{\mathbb{C}T} \circ d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})_u^{-1}(X_{-2}) \\
& & \wedge\big(d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})_u^{-1}(X_{11})\big) \\
& & \wedge \cdots \wedge \big(d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})_u^{-1}(X_{1n})\big)
\end{array}
$$

As we have seen in Subsection 4.2.2 P acts on $\mathfrak{g}^{\mathbb{C}}_{-2}\wedge\mathfrak{g}_{10}^{\wedge n}$ in the following way. For an element $A = \begin{pmatrix} a & -aZ^*B & b \\ 0 & B & Z \\ 0 & 0 & \overline{a}^{-1} \end{pmatrix} \in P$ we have $\widetilde{Ad}^*(A) = a^{-(n+2)}$.

For the homogeneous case we have accordingly $\mathcal{K}^* \simeq \mathcal{C} \times_{[\mathbb{C}^*, z^{-n-2}]} \mathbb{C}^*$ as we have seen in Section 5.3 .

We now check whether the isomorphism Iso_4^* is well-defined, that is to say independent of the representative of the class $[u, X_{-2} \wedge X_{11} \wedge \cdots \wedge X_{1n}]$. The representative

$$\big(R_p u, Ad^{\mathbb{C}}_{\mathfrak{g}_{-2}}(p^{-1})X_{-2} \wedge Ad^{\mathbb{C}}_{\mathfrak{g}_{-1}}(p^{-1})X_{11} \wedge \cdots \wedge Ad^{\mathbb{C}}_{\mathfrak{g}_{-1}}(p^{-1})X_{1n}\big) \in [u, X_{-2} \wedge X_{11} \wedge \cdots \wedge X_{1n}]$$

is mapped to:

$$
\begin{array}{l}
pr_{\mathbb{C}T} \circ d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})^{-1}_{R_p u}\big(Ad^{\mathbb{C}}_{\mathfrak{g}_{-2}}(p^{-1})X_{-2}\big) \\
\wedge\big(d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})^{-1}_{R_p u}\big(pr_{\mathfrak{g}^{\mathbb{C}}_{-1}} \circ \underbrace{Ad(p^{-1})X_{11}}_{\in \mathfrak{g}^{\mathbb{C}}_{-1}+\mathfrak{p}^{\mathbb{C}}}\big)\big) \\
\wedge \cdots \wedge \big(d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})^{-1}_{R_p u}\big(pr_{\mathfrak{g}^{\mathbb{C}}_{-1}} \circ \underbrace{Ad(p^{-1})X_{1n}}_{\in \mathfrak{g}^{\mathbb{C}}_{-1}+\mathfrak{p}^{\mathbb{C}}}\big)\big).
\end{array}
$$

Due to the projection $d\pi^{\mathbb{C}}$ we can neglect the components of $\mathfrak{p}^{\mathbb{C}}$ in our calculations and continue computing the image as

$$
\begin{array}{rl}
& pr_{\mathbb{C}T} \circ d\underline{\pi}^{\mathbb{C}} \circ \underbrace{dR_p \circ (\underline{\omega}^{\mathbb{C}})_u^{-1} \circ Ad(p)}_{=(\underline{\omega}^{\mathbb{C}})^{-1}_{R_p u}} \big(Ad(p^{-1})X_{-2} - pr_{\mathfrak{g}^{\mathbb{C}}_{-1}} \circ Ad(p^{-1})X_{-2}\big) \\
& \wedge\big(d\underline{\pi}^{\mathbb{C}} \circ \underbrace{dR_p \circ (\underline{\omega}^{\mathbb{C}})_u^{-1} \circ Ad(p)}_{=(\underline{\omega}^{\mathbb{C}})^{-1}_{R_p u}} \big(Ad(p^{-1})X_{11}\big)\big) \\
& \wedge \cdots \wedge \big(d\underline{\pi}^{\mathbb{C}} \circ \underbrace{dR_p \circ (\underline{\omega}^{\mathbb{C}})_u^{-1} \circ Ad(p)}_{=(\underline{\omega}^{\mathbb{C}})^{-1}_{R_p u}} \big(Ad(p^{-1})X_{1n}\big)\big) \\
= & \Big(\; pr_{\mathbb{C}T} \circ d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})_u^{-1}(X_{-2}) - pr_{\mathbb{C}T} \circ \underbrace{d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})_u^{-1} \circ Ad(p) \circ pr_{\mathfrak{g}^{\mathbb{C}}_{-1}} \circ Ad(p^{-1})(X_{-2})}_{\in T^{-1}M^{\mathbb{C}}} \; \Big) \\
& \wedge\big(d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})_u^{-1}(X_{11})\big) \wedge \cdots \wedge \big(d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})_u^{-1}(X_{1n})\big) \\
= & pr_{\mathbb{C}T} \circ d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})_u^{-1}(X_{-2}) \wedge \big(d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})_u^{-1}(X_{11})\big) \wedge \cdots \wedge \big(d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})_u^{-1}(X_{1n})\big).
\end{array}
$$

So the isomorphism Iso_4^* is well-defined.

The action of $\underline{P}$ on $\mathfrak{g}^{\mathbb{C}}_{-2} \wedge \mathfrak{g}_{10}^{\wedge n}$ by $\widetilde{Ad}^*$ is equivalent to $\underline{P}$ acting on the complex null line $\ell = \mathbb{C}f_-$ (see Section 4.1) by

$$
\begin{array}{rcl}
\varrho : \underline{P} \times \ell & \longrightarrow & \ell \\
(A, X) & \mapsto & A^{-n-2}X = a^{-n-2}X.
\end{array}
$$

Hence we can write $Iso_5 : \mathcal{K}^* \simeq \underline{\mathcal{G}} \times_{[\underline{P},\varrho]} \ell$.

And we obtain for the frame bundle of $\mathcal{K}^*$

$$Iso_6 : \mathcal{F}(\mathcal{K}^*) = (\mathcal{K}^* \setminus \{0\})/_{\mathbb{R}^+} \simeq \underline{\mathcal{G}} \times_{[\underline{P},\varrho]} (\ell \setminus \{0\})/_{\mathbb{R}^+}.$$

Note that the action ϱ is transitive on $\ell\backslash\{0\}$ and therefore transitive on the rays $(\ell\backslash\{0\})/_{\mathbb{R}^+}$. Recall the denotations of Section 5.2. G is the special unitary group $G = SU(1, n+1)$, P denotes the stabilizer of a complex line $P = stab_G(\underbrace{\ell}_{=\mathbb{C}f_-})$. Working modulo the center of G is marked by underlining, $\underline{G} = G/_{Z(G)}$ and $\underline{P} = stab_{\underline{G}}(\mathbb{C}f_-) = P/_{Z(G)}$. $\widetilde{G}$ is the connected component of the identity in the orthogonal group $O_c(\mathbb{C}^{1,n+1}, \langle\cdot,\cdot\rangle_{\mathbb{R}})$ and $\widetilde{P}$ is the stabilizer of the real line $\widetilde{\ell} = \mathbb{R}f_-$, $\widetilde{P} = stab_{\widetilde{G}}(\widetilde{\ell})$. The stabilizer of the real ray $\mathbb{R}^+f_- \subset \ell\setminus\{0\}$ with respect to the action ϱ is $(\widetilde{P}^+ \cap G)\cdot Z(G)$, where $\widetilde{P}^+$ denotes the positive part of $\widetilde{P}$, i.e.

$$\widetilde{P}^+ := \big\{A \in \widetilde{G} \;\big|\; Af_- = \lambda f_- \text{ for a } \lambda \in \mathbb{R}^+\big\}.$$

Obviously it holds $(\widetilde{P}^+ \cap G)\cdot Z(G) \subset \underline{P}$ since we already know that $\widetilde{P}\cap G \subset P$ holds.
So we can write

$$\begin{array}{rcl} Iso_7 : \mathcal{F}(\mathcal{K}^*) \simeq \underline{\mathcal{G}} \times_{[\underline{P},\varrho]} (\ell\setminus\{0\})/_{\mathbb{R}^+} & \xrightarrow{\sim} & \underline{\mathcal{G}}/_{(\widetilde{P}^+\cap G)\cdot Z(G)} \\ {[u,[e]] = [R_A u, [f_-]]} & \mapsto & [R_A u]. \end{array}$$

With this isomorphism $\mathcal{F}(\mathcal{K}^*)$ inherits the right action of $\underline{P}/_{(\widetilde{P}^+\cap G)\cdot Z(G)}$ on $\underline{\mathcal{G}}/_{(\widetilde{P}^+\cap G)\cdot Z(G)}$. Let us therefore take a closer look at the group $\widetilde{P}^+ \cap G \subset P$. We have

$$\begin{array}{rcl} \widetilde{P}^+ \cap G & = & \{A \in G \mid Af_- = \lambda f_- \text{ for a } \lambda \in \mathbb{R}\} \\ & & \text{and with } \widetilde{P}^+\cap G \subset P \text{ we can write} \\ & = & \{A \in P \mid Af_- = \lambda f_- \text{ for a } \lambda \in \mathbb{R}\} \\ & = & \left\{ \begin{pmatrix} a & -a\bar{z}^t B & b \\ 0 & B & z \\ 0 & 0 & a^{-1} \end{pmatrix} \;\middle|\; \begin{array}{l} a \in \mathbb{R}^+, b \in \mathbb{C}, B \in U(n), z \in \mathbb{C}^n \\ det(B) = 1 \\ \frac{b+\bar{b}}{a} + \|z\|^2 = 0 \end{array} \right\}. \end{array}$$

Note that $\widetilde{P}^+ \cap G \subset P$ is a normal subgroup in P. With the homomorphism

$$\begin{array}{rcl} \rho : S^1 & \longrightarrow & Aut(\widetilde{P}\cap G) \\ e^{i\varphi} & \mapsto & L_{M(e^{i\varphi})} \circ R_{M(e^{i\varphi})^{-1}}, \\ & & \text{where } M(e^{i\varphi}) := \begin{pmatrix} e^{i\varphi} & 0 & 0 \\ 0 & e^{-\frac{2i\varphi}{n}} I_n & 0 \\ 0 & 0 & e^{i\varphi} \end{pmatrix}, \end{array}$$

we can form the semi-direct product $S^1 \ltimes_\rho (\widetilde{P}^+ \cap G)$ equipped with the multiplication $(e^{i\varphi_1}, \widetilde{A}_1)\cdot(e^{i\varphi_2}, \widetilde{A}_2) = \Big(e^{i\varphi_1}\cdot e^{i\varphi_2}, \rho(e^{-i\varphi_2})(\widetilde{A}_1)\cdot \widetilde{A}_2\Big)$.
Hence we can see P as the semi-direct product of S^1 and $\widetilde{P}^+ \cap G$,

$$\begin{array}{rcl} S^1 \ltimes_\rho (\widetilde{P}^+ \cap G) & \longrightarrow & P \\ (e^{i\varphi}, \widetilde{A}) & \mapsto & M(e^{i\varphi})\cdot \widetilde{A}. \end{array}$$

So with the two projections

$$\begin{array}{rrcll} & pr_{S^1} : P & \longrightarrow & P/_{(\widetilde{P}^+\cap G)} & \simeq S^1 \\ \text{and} & \hat{\pi} : \underline{\mathcal{G}} & \longrightarrow & \underline{\mathcal{G}}/_{(\widetilde{P}^+\cap G)\cdot Z(G)} & \simeq \mathcal{F}(\mathcal{K}^*) \end{array}$$

and interpreting the center of G as a subgroup of the sphere S^1 we can write

$$\hat{\pi}\left(R_{A\cdot Z(G)} u\right) = R_{pr_{S^1}(A)\cdot Z(G)} \circ \hat{\pi}(u) \text{ for all } u \in \underline{\mathcal{G}} \text{ and all } A \in P.$$

As we have seen this action corresponds to multiplication by $a^{-(n+2)}$ for a being the first entry of the first row of the matrix $A \in P$.

We have required the existence of a complex line bundle $\mathcal{E}(1,0) \longrightarrow M$ together with a duality between $\mathcal{E}(1,0)^{\otimes(n+2)}$ and the canonical complex line bundle of M. So by definition we have $\mathcal{E}(1,0)^{\otimes(n+2)} = \mathcal{E}(1,0) \times_{[\mathbb{C}^*,\lambda^{n+2}]} \mathbb{C}$. We further set $\mathcal{E}(-1,0) := \mathcal{E}(1,0) \times_{[\mathbb{C}^*,\lambda^{-1}]} \mathbb{C}$. With these we obtain a $(n+2)$-fold covering

$$\begin{array}{ccc} \mathcal{F}(\mathcal{E}(-1,0)) = \mathcal{E}(1,0) \times_{[S^1,\lambda^{-1}]} S^1 & \xrightarrow{\phi} & \mathcal{F}(\mathcal{K}^*) \simeq \mathcal{F}(\mathcal{E}(1,0)^{\otimes(n+2)}) = \mathcal{E}(1,0) \times_{[S^1,\lambda^{n+2}]} S^1 \\ [u,\varphi] = [R_\lambda u, \lambda\varphi] & \mapsto & [u, \varphi^{-(n+2)}] = [R_\lambda u, \lambda^{-(n+2)}\varphi^{-(n+2)}]. \end{array}$$

Note that ϕ is compatible with the right actions of S^1 resp. $S^1 \cdot Z(G)$, $\phi \circ R_\lambda = R_{\lambda \cdot Z(G)} \circ \phi$. Now we can pull back the $(n+2)$-fold covering to $\underline{\mathcal{G}}$.

$$\begin{array}{ccccc} & & & & \mathcal{F}\big(\mathcal{E}(-1,0)\big) \\ & & & & \downarrow \phi \ \ {\scriptstyle (n+2):1} \\ \underline{\mathcal{G}} & \xrightarrow{\hat{\pi}} & \underline{\mathcal{G}}/_{(\widetilde{P}^+ \cap G)\cdot Z(G)} & \overset{Iso_7}{\simeq} & \mathcal{F}(\mathcal{K}^*) \end{array}$$

This results in $\mathcal{G} = \hat{\pi}^* \mathcal{F}\big(\mathcal{E}(-1,0)\big)$, a $(n+2)$-fold covering of $\underline{\mathcal{G}}$,

$$\mathcal{G} := \hat{\pi}^* \mathcal{F}\big(\mathcal{E}(-1,0)\big) = \{(\underline{u}, x) \in \underline{\mathcal{G}} \times \mathcal{F}\big(\mathcal{E}(-1,0)\big) \mid \hat{\pi}(\underline{u}) = \phi(x)\}.$$

Actually $\hat{\pi}(u) = \phi(x)$ means $Iso_7 \circ \hat{\pi}(u) = \phi(x)$. However we will neglect writing Iso_7 in order to simplify notations.

$$\begin{array}{ccccccc} \underline{\mathcal{G}} \times \mathcal{F}\big(\mathcal{E}(1,0)\big) & \supset & \mathcal{G} & \xrightarrow{pr_2} & \mathcal{F}\big(\mathcal{E}(1,0)\big) & & \\ & pr_1 & \downarrow & & \downarrow \phi \ \ {\scriptstyle (n+2):1} & & \\ & & \underline{\mathcal{G}} & \xrightarrow{\hat{\pi}} & \underline{\mathcal{G}}/_{(\widetilde{P}^+ \cap G)\cdot Z(G)} & \overset{Iso_7}{\simeq} & \mathcal{F}(\mathcal{K}^*). \end{array}$$

We need to lift the right action of $\underline{P}$ on $\underline{\mathcal{G}}$ to a right action of P on $\mathcal{G}$. We set

$$\begin{array}{ccc} \mathcal{G} \times P & \longrightarrow & \mathcal{G} \\ \big((\underline{u}, x), A\big) & \mapsto & \big(R_{A\cdot Z(G)}\underline{u}, R_{pr_{S^1}A}x\big). \end{array}$$

This is well-defined since we have for all $A \in P$

$$\begin{aligned} \hat{\pi}(R_{A\cdot Z(G)}\underline{u}) &= R_{pr_{S^1}(A)\cdot Z(G)} \circ \hat{\pi}(\underline{u}) \\ &= R_{pr_{S^1}(A)\cdot Z(G)} \circ \phi(x) \\ &= \phi(R_{pr_{S^1}(A)}x). \end{aligned}$$

The Cartan connection on $\mathcal{G}$ is obtained by pulling back $\underline{\omega}$, $\omega := pr_1^* \underline{\omega}$. This is well-defined since devision by the centre of G has no influence on the Lie algebras and the Adjoint action of $Z(G) = \{e^{\frac{2\pi i k}{n+2}} I_{n+2} \mid k = 0, \ldots, n+1\}$ on $\mathfrak{g}$ is trivial. So all the properties required are directly inherited from $\underline{\omega}$, that is to say ω is P-equivariant, reproduces the generators of the fundamental vector fields and defines a trivialization of the tangent space of $\mathcal{G}$.
Analogously to the homogeneous case we now define the Fefferman space

$$\mathcal{F} := \mathcal{G}/_{\widetilde{P} \cap G}.$$

Consequently we can view the total space of the Cartan bundle of the CR-manifold M as a principal $G \cap \widetilde{P}$-bundle over the Fefferman space $\mathcal{F}$,

$$\mathcal{G} \xrightarrow{\pi_{\mathcal{F}}} \mathcal{F} \text{ with structure group } G \cap \widetilde{P}.$$

Projecting $\widetilde{P} \cap G$ to S^1 gives $\{\pm 1\}$. So we can write

$$\begin{aligned} \mathcal{F} &= \mathcal{G}/_{\widetilde{P} \cap G} \\ &= \mathcal{G}/_{\widetilde{P}^+ \cap G} \Big/_{\{\pm 1\}} \\ &= \Big\{ (\underline{u}, x) \in \underline{\mathcal{G}}/_{(\widetilde{P}^+ \cap G)\cdot Z(G)} \times \mathcal{F}(\mathcal{E}(-1,0))/_{\{\pm 1\}} \ \Big| \ \underline{u} = \phi(x) \Big\} \\ &= \mathcal{F}\big(\mathcal{E}(-1,0)\big)/_{\{\pm 1\}} \\ &= \big(\mathcal{E}(-1,0) \setminus \{0\}\big)/_{\mathbb{R}^*}. \end{aligned}$$

On the other hand we can write

$$\begin{aligned}\mathcal{F} &= \mathcal{G}/_{\widetilde{P}\cap G} \\ &= \mathcal{G}\times_P P/_{\widetilde{P}\cap G}.\end{aligned}$$

As we have seen P acts transitively on the complex null line ℓ and in G the stabilizer of the real null line $\widetilde{l}$ is $\widetilde{P}\cap G$. So we obtain

$$\begin{aligned}\mathcal{F} &= \mathcal{G}\times_P P/_{\widetilde{P}\cap G} \\ &= \mathcal{G}\times_P (\ell\setminus\{0\})/_{\mathbb{R}^*}.\end{aligned}$$

It remains to construct the conformal class of the Fefferman space. The Cartan connection $\omega_M \in \Omega^1(\mathcal{G},\mathfrak{g})$ of the Cartan bundle of the CR-manifold is also a Cartan connection for the $G\cap\widetilde{P}$-principal bundle $\mathcal{G}\longrightarrow\mathcal{F}$ since it is automatically $G\cap\widetilde{P}$-equivariant and reproduces the generators of the fundamental vector fields $\widetilde{X}$ with $X\in\mathfrak{g}\cap\widetilde{\mathfrak{p}}$, as both properties are restrictions of the corresponding properties satisfied by ω_M regarded as a Cartan connection for the CR-manifold M. And of course ω_u is also a linear isomorphism, $\omega_u : T_u\mathcal{G}\longrightarrow\mathfrak{g}$, for every $u\in\mathcal{G}$.
So in the usual way the tangent space of the Fefferman space can be written as an associated vector bundle:

$$\begin{array}{rcll} T\mathcal{F} & \longrightarrow & \mathcal{G}\times_{G\cap\widetilde{P}}\mathfrak{g}/_{\mathfrak{g}\cap\widetilde{\mathfrak{p}}} & \\ X_x & \rightarrowtail & \big[u,\omega_u(\overline{X})+\mathfrak{g}\cap\widetilde{\mathfrak{p}}\big] & \begin{array}{l}\text{with } u\in\mathcal{G},\pi_{\mathcal{F}}(u)=x\in\mathcal{F}\\ \text{and } \overline{X}\in T\mathcal{G}, d\pi_{\mathcal{F}}\overline{X}=X\in T\mathcal{F}\end{array} \\ d\pi_{\mathcal{F}}\circ\omega_u^{-1}(v) & \longleftarrow & [u,v] & \end{array}.$$

Now we can use the in [Feh05] described natural, oriented, conformal structure $c^{M\ddot{o}b}$ of the Möbius space $\widetilde{G}/_{\widetilde{P}}\simeq G/_{G\cap\widetilde{P}}$ to define the conformal structure on $\mathcal{F}$. The isomorphism $\widetilde{G}/_{\widetilde{P}}\simeq G/_{G\cap\widetilde{P}}$ and thus $\widetilde{\mathfrak{g}}/_{\widetilde{\mathfrak{p}}}\simeq\mathfrak{g}/_{\mathfrak{g}\cap\widetilde{\mathfrak{p}}}$ is explained in Section 5.2. Thus for $X_x\in T_x\mathcal{F}$ and $\overline{X}\in T_u\mathcal{G}$ with $d\pi_{\mathcal{F}}\overline{X}=X_x$ we have $\omega_u(\overline{X})+\mathfrak{g}\cap\widetilde{\mathfrak{p}}\in\mathfrak{g}/_{\mathfrak{g}\cap\widetilde{\mathfrak{p}}}\simeq\widetilde{\mathfrak{g}}/_{\widetilde{\mathfrak{p}}}=T_{e\widetilde{P}}\widetilde{G}/_{\widetilde{P}}$ and this is independent of the lift $\overline{X}$ chosen. With $g_{\text{Möb}}\in c^{M\ddot{o}b}$ we set

$$g(X_x,Y_x) := g_{\text{Möb}}\big(\omega_u(\overline{X})+\mathfrak{g}\cap\widetilde{\mathfrak{p}},\omega_u(\overline{Y})+\mathfrak{g}\cap\widetilde{\mathfrak{p}}\big),$$

and define finally $[g]$ to be the conformal class of g on the Fefferman space $\mathcal{F}$. Since we started of with a strictly pseudo-convex CR manifold of real dimension $2n+1$, i.e. $p=0, q=n$, the signature of the Fefferman space is $(1,2n+1)$, although in many calculations we use the arbitrary signature $G=SU(p+1,q+1)$ to indicate, that in many cases the calculations do not depend on the signature.
The canonical Cartan connections ω_M and $\omega_{\mathcal{F}}$ are characterized by the following properties:

- ω_M has ∂^*-closed curvature, that is $\partial^*\circ\Omega^{\omega_M}=0$.
- The homogeneous components of the curvature of degree less or equal to zero vanish, $(\Omega^{\omega_M})^l\equiv 0$ for $l\leq 0$.
- ω_M is torsion free, that is to say $t := pr_{\mathfrak{g}_-}\circ\Omega^{\omega_M}=0$.
- $\omega_{\mathcal{F}}$ is admissible, i.e. $\omega_{\mathcal{F}}=\theta_{\mathcal{G}_{\mathcal{F}}}\oplus\omega_{\mathfrak{g}_1}$, where $\theta_{\mathcal{G}_{\mathcal{F}}}$ denotes the displacement form of $\mathcal{G}_{\mathcal{F}}$.
- The torsion of $\omega_{\mathcal{F}}$ is ∂^*-closed which is equivalent to $\Omega_0^{\omega_{\mathcal{F}}}$ being ∂^*-closed.

Remark 5.2 *Please note that we have the following correspondence between the Fefferman space constructed according to [BL04] and the construction suggested by [CG08].* $\mathcal{E}(1,0)\times_{[S^1,\lambda^{-1}]} S^1$ *is a two fold covering of* $\mathcal{F}_{[CG08]}$

$$\begin{array}{rcccl} \mathcal{E}(1,0)\times_{[S^1,\lambda^{-1}]} S^1 & \longrightarrow & \mathcal{F}_{[CG08]} \simeq \left(\mathcal{E}(1,0)\times_{[S^1,\lambda^{-1}]}\right)/_{\{\pm 1\}} & \simeq & \mathcal{E}(1,0)\times_{[S^1,\lambda^{-2}]} S^1 \\ {[u,\varphi]} = [R_\lambda u, \lambda\varphi] & \mapsto & [u,\pm\varphi] = [R_\lambda u, \pm\lambda\varphi] & \mapsto & [u,\varphi^2] = [R_\lambda u, \lambda^2\varphi^2]. \end{array}$$

And $\mathcal{E}(1,0)\times_{[S^1,\lambda^{-1}]} S^1$ *is a* $(n+2)$ *fold covering of* $\mathcal{F}_{[BL04]}$

$$\begin{array}{rcl} \mathcal{E}(1,0)\times_{[S^1,\lambda^{-1}]} S^1 & \longrightarrow & \mathcal{F}_{[BL04]} \simeq \mathcal{E}(1,0)\times_{[S^1,\lambda^{n+2}]} S^1 \\ {[u,\varphi]} = [R_\lambda u, \lambda\varphi] & \mapsto & [u,\varphi^{-(n+2)}] = [R_\lambda u, \lambda^{-(n+2)}\varphi^{-(n+2)}]. \end{array}$$

5.5 The Cartan Bundle of the Fefferman Space

The standard tractor bundle of the CR manifold $(M,H,\theta,\mathcal{E}(1,0))$ is defined as the associated vector bundle $\mathcal{T} := \mathcal{G}\times_P \mathbb{C}^{1,n+1}$. This is endowed with a hermitian product h of signature $(1,n+1)$ induced by the hermitian product $\langle\cdot,\cdot\rangle_{1,n+1}$ of $\mathbb{C}^{1,n+1}$,

$$\begin{aligned} h\big([u,X],[u,Y]\big) &:= \langle X,Y\rangle_{1,n+1} \\ &= \langle pX,pY\rangle_{1,n+1} \text{ for all } p\in P. \end{aligned}$$

The P-invariant complex line $\mathbb{C}f_-$ defines a natural subbundle $\mathcal{T}^1 := \mathcal{G}\times_P \mathbb{C}f_-$. Its fibres are null with respect to h.

We furthermore obtain another associated vector bundle over the Fefferman space $\mathcal{F}$ via $\widetilde{\mathcal{T}} := \mathcal{G}\times_{G\cap\widetilde{P}} \mathbb{C}^{1,n+1} \longrightarrow \mathcal{F}$. This bundle is equipped with a real bundle metric $\widetilde{g}\big([u,X],[u,Y]\big) := Re\langle X,Y\rangle_{1,n+1}$ of signature $(2,2n+2)$. The real line $\mathbb{R}f_-$ stabilized by $G\cap\widetilde{P}$ gives rise to a real line subbundle $\widetilde{\mathcal{T}}^1 := \mathcal{G}\times_{G\cap\widetilde{P}} \mathbb{R}f_-$ whose fibres are null with respect to $\widetilde{g}$. We denote the orthogonal complement of this line bundle with $\widetilde{\mathcal{T}}^0 := \left(\widetilde{\mathcal{T}}^1\right)^{\perp_{\widetilde{g}}}$ and obtain the filtration

$$\widetilde{\mathcal{T}} =: \widetilde{\mathcal{T}}^{-1} \supset \widetilde{\mathcal{T}}^0 \supset \widetilde{\mathcal{T}}^1.$$

The Cartan connection ω of the CR manifold M induces a tractor connection $\nabla^{\widetilde{\mathcal{T}}}$.

Proposition 5.1 *Let* $(M,H,\theta,\mathcal{E}(1,0))$ *be a strictly pseudo-convex CR manifold of dimension* $2n+1$ *together with an* $(n+2)$*nd root of the anticanonical complex line bundle. Then* $\left(\widetilde{\mathcal{T}},\widetilde{\mathcal{T}}^1,\widetilde{g},\nabla^{\widetilde{\mathcal{T}}}\right)$ *is a standard tractor bundle for* $(\mathcal{F},[h_\theta])$ *and* $\nabla^{\widetilde{\mathcal{T}}}$ *is normal. Furthermore the Cartan bundle of the Fefferman Space is*

$$\widetilde{\mathcal{G}} = \mathcal{G}\times_{G\cap\widetilde{P}} \widetilde{P}$$

and its Cartan connection is given as

$$\widetilde{\omega}_{[u,\widetilde{p}]} = Ad(\widetilde{p}^{-1})\circ\pi_{\mathcal{G}}^*\omega + \pi_{\widetilde{P}}^*\omega_{\widetilde{P}}.$$

Proof: For the prove of this proposition we will use Proposition 3.9 from Subsection 3.6.2. First of all $\pi : \widetilde{\mathcal{T}} \longrightarrow \mathcal{F}$ is a vector bundle of real rank $2n+4$ endowed with a bundle metric $\widetilde{g}$ of signature $(2,2n+2)$. The fibres of the subbundle $\widetilde{\mathcal{T}}^1$ are null with respect to $\widetilde{g}$.

The subgroup $\widetilde{P}$ acts on the $(2n+2)$nd tensor power of the real null line ℓ by multiplication with the $(2n+2)$nd power of the conformal factor, that means for $\widetilde{p} = \begin{pmatrix} a & * & * \\ 0 & A & * \\ 0 & 0 & a^{-1} \end{pmatrix}$ we have $\varrho(\widetilde{p})\left(\lambda f_-^{\otimes(2n+2)}\right) = a^{2n+2}\lambda f_-^{\otimes(2n+2)}$. As can be found in [Feh05] $\widetilde{P}$ acts on $\widetilde{\mathfrak{g}}/_{\widetilde{\mathfrak{p}}}$ via $Ad(\widetilde{p})(v) = a^{-1}Av$. Thus its action on the $(2n+2)$nd exterior power is dual to the action on $\ell^{\otimes(2n+2)}$, that is to say for $\theta \in \left(\widetilde{\mathfrak{g}}/_{\widetilde{\mathfrak{p}}}\right)^{\wedge(2n+2)}$ it holds

$$Ad^{\wedge(2n+2)}(\widetilde{p})(\theta) = a^{-(2n+2)} \underbrace{\det(A)}_{=1} \theta = a^{-(2n+2)}\theta.$$

However this is the dual of the basic density bundle $\mathcal{E}_{\mathcal{F}}[-n] = \mathcal{Q}_{\mathcal{F}} \times_{[\mathbb{R}_+, s^n]} \mathbb{R}$ and therefore the null-line bundle $\widetilde{\mathcal{T}}^1$ is isomorphic to the density bundle $\mathcal{E}_{\mathcal{F}}[-1]$.
The Cartan connection ω of the Cartan bundle $\mathcal{G} \longrightarrow \mathcal{F}$ induces according to Proposition 3.7 a tractor connection $\nabla^{\widetilde{\mathcal{T}}}$ on the tractor bundle $\widetilde{T} = \mathcal{G} \times_{G\cap\widetilde{P}} \mathbb{C}^{1,n+1}$, i.e. $\nabla^{\widetilde{\mathcal{T}}}$ is a nondegenerate $\mathfrak{o}(2, 2n+2)$-connection, which is according to Subsection 3.6.2 equivalent to $\nabla^{\widetilde{T}}$ being compatible with the bundle metric $\widetilde{g}$ and that we can find for every point $x \in \mathcal{F}$ and every vector $\xi \in T_x\mathcal{F}$ a section $\sigma \in \Gamma(\widetilde{\mathcal{T}}^1)$ such that $\nabla^{\widetilde{\mathcal{T}}}_{\xi}\sigma \notin \widetilde{\mathcal{T}}^1_x$.
Thus according to Proposition 3.9 $(\widetilde{\mathcal{T}}^1)^{\perp}/_{\widetilde{\mathcal{T}}^1}$ is isomorphic to $T\mathcal{F} \otimes \mathcal{E}[-1]$ and $(\widetilde{\mathcal{T}}, \widetilde{g}, \nabla^{\widetilde{\mathcal{T}}})$ is a standard tractor bundle for $\mathcal{F}$ endowed with the conformal structure defined by the restriction of the bundle metric, $\widetilde{g}\big|_{(\widetilde{\mathcal{T}}^1)^{\perp}/_{\widetilde{\mathcal{T}}^1} \times (\widetilde{\mathcal{T}}^1)^{\perp}/_{\widetilde{\mathcal{T}}^1}}$. Hence we need to prove that both conformal structures coincide.
For vectors $X, Y \in T_x\mathcal{F}$ and a section $\varphi \in \Gamma(\widetilde{\mathcal{T}}^1) \simeq \Gamma(\mathcal{E}[-1])$ we define in compliance with Subsection 3.6.2

$$\widetilde{g}_{\varphi}(X,Y) \;=\; \widetilde{g}(\nabla^{\widetilde{\mathcal{T}}}_X\varphi + \widetilde{\mathcal{T}}^1, \nabla^{\widetilde{\mathcal{T}}}_Y\varphi + \widetilde{\mathcal{T}}^1).$$

Please recall that $\nabla^{\widetilde{\mathcal{T}}}_X\varphi$ is a section in $\widetilde{\mathcal{T}}^0$ and can be expressed in terms of the Cartan connection as $\nabla^{\widetilde{\mathcal{T}}}_X\varphi = \left[u, \overline{X}(\widetilde{\varphi})_u + \varrho(\omega(\overline{X}))(\widetilde{\varphi})_u\right]$, where $\overline{X} \in T_u\mathcal{G}$ is a lift of $X \in T_x\mathcal{F}$ and $\widetilde{\varphi} : \mathcal{G} \longrightarrow \ell \subset \mathbb{C}^{1,n+1}$ is the to φ corresponding $G \cap \widetilde{P}$-equivariant map. Thus we can write

$$\widetilde{g}_{\varphi}(X,Y) \;=\; \langle \underbrace{\overline{X}(\widetilde{\varphi})_u}_{\in \ell = span(f_-)} + \varrho(\omega(\overline{X}))(\widetilde{\varphi})_u + \ell, \overline{Y}(\widetilde{\varphi})_u + \varrho(\omega(\overline{Y}))(\widetilde{\varphi})_u + \ell\rangle_{2,2n+2}.$$

However we have $\varrho(\omega(\overline{X}))(\widetilde{\varphi})_u \in span(f_-, if_-, e_1, ie_1, \ldots, e_n, ie_n, if_+) = \left(span(f_-)\right)^{\perp}$. So we can simplify the equation above.

$$\widetilde{g}_{\varphi}(X,Y) \;=\; \langle \varrho(\omega(\overline{X}))(\widetilde{\varphi})_u, \varrho(\omega(\overline{Y}))(\widetilde{\varphi})_u\rangle_{2,2n+2}$$

And with $\omega(\overline{X}) = \begin{pmatrix} z(\overline{X}) & * & * \\ Z(\overline{X}) & * & * \\ ia(\overline{X}) & * & * \end{pmatrix}$ we can write with $c \in \mathbb{R}$ a factor depending on φ

$$\begin{aligned} \widetilde{g}_{\varphi}(X,Y) \;&=\; c^2 \left\langle \begin{pmatrix} z(\overline{X}) \\ Z(\overline{X}) \\ ia(\overline{X}) \end{pmatrix}, \begin{pmatrix} z(\overline{Y}) \\ Z(\overline{Y}) \\ ia(\overline{Y}) \end{pmatrix} \right\rangle_{2,2n+2} \\ &=\; c^2 \left\langle \begin{pmatrix} iIm\big(z(\overline{X})\big) \\ Z(\overline{X}) \\ ia(\overline{X}) \end{pmatrix}, \begin{pmatrix} iIm\big(z(\overline{Y})\big) \\ Z(\overline{Y}) \\ ia(\overline{Y}) \end{pmatrix} \right\rangle_{2,2n+2}. \end{aligned}$$

On the other hand the Fefferman space was endowed with the conformal class inherited from the Möbius space. With the help of a section $\mu : \mathbb{PC} \simeq G/_{G\cap\widetilde{P}} \longrightarrow \mathcal{C}^+ \subset \mathbb{R}^{2,2n+2}$ we define

$$\begin{aligned}
g_\mu(X,Y) &= g_\mu^{\text{Möb}}\big(\omega_u(\overline{X}) + \mathfrak{g}\cap\widetilde{\mathfrak{p}}, \omega_u(\overline{Y}) + \mathfrak{g}\cap\widetilde{\mathfrak{p}}\big) \\
&= \big\langle d\mu\big(\omega_u(\overline{X}) + \mathfrak{g}\cap\widetilde{\mathfrak{p}}\big), d\mu\big(\omega_u(\overline{Y}) + \mathfrak{g}\cap\widetilde{\mathfrak{p}}\big)\big\rangle_{2,2n+2} \\
&= c^2(\mu)\big\langle \omega_u(\overline{X}) + \mathfrak{g}\cap\widetilde{\mathfrak{p}}, \omega_u(\overline{Y}) + \mathfrak{g}\cap\widetilde{\mathfrak{p}}\big\rangle_{2,2n+2} \\
&= c^2(\mu)\left\langle \begin{pmatrix} iIm\big(z(\overline{X})\big) \\ Z(\overline{X}) \\ ia(\overline{X}) \end{pmatrix}, \begin{pmatrix} iIm\big(z(\overline{Y})\big) \\ Z(\overline{Y}) \\ ia(\overline{Y}) \end{pmatrix} \right\rangle_{2,2n+2}.
\end{aligned}$$

Thus the conformal structures defined by g_μ and by $\widetilde{g}_\varphi$ are the same and $(\widetilde{\mathcal{T}}, \widetilde{g}, \nabla^{\widetilde{\mathcal{T}}})$ is a standard tractor bundle of the Fefferman space. It remains to check the normality conditions. This can be done with the help of the Cartan bundle of the Fefferman space. As we have seen in Subsection 3.6.2 the Cartan bundle of the Fefferman space $\mathcal{F}$ can be recovered from the tractor bundle $\widetilde{\mathcal{T}}$.

$$\widetilde{\mathcal{G}} = \big\{\gamma_x : \mathbb{C}^{1,n+1} \longrightarrow \widetilde{\mathcal{T}}_x \;\big|\; x \in \mathcal{F}, \gamma_x \text{ orthogonal}, \gamma(\ell) \subset \widetilde{\mathcal{T}}^1_x\big\}$$

This is isomorphic to the $\widetilde{P}$-principal bundle $\mathcal{G} \times_{G\cap\widetilde{P}} \widetilde{P}$ since

$$\begin{array}{rcl}
\widetilde{\mathcal{G}} & \longrightarrow & \mathcal{G} \times_{G\cap\widetilde{P}} \widetilde{P} \\
\left.\begin{array}{r} \gamma_x \\ \text{with } \gamma_x(v) = \underbrace{[u, A^u v]}_{\in \widetilde{\mathcal{T}} = \mathcal{G}\times_{G\cap\widetilde{P}}\mathbb{C}^{1,n+1}} = [R_p u, \underbrace{p^{-1}\cdot A^u}_{=A^{R_p u}} v] \\ \text{and } A^u \in \widetilde{P} \end{array}\right\} & \mapsto & [u, A^u] = [R_p u, p^{-1}\cdot A^u] = [R_p u, A^{R_p u}] \\
R_p \gamma_x = \gamma_x \circ p & \mapsto & [u, A\cdot p] = R_p[u, A].
\end{array}$$

Thus sections of the tractor bundle $\widetilde{\mathcal{T}}$ can be written as $G\cap\widetilde{P}$-equivariant maps $\mathcal{G} \longrightarrow \mathbb{C}^{1,n+1}$ or as $\widetilde{P}$-equivariant maps $\widetilde{\mathcal{G}} \longrightarrow \mathbb{C}^{1,n+1}$.

$$\begin{array}{rcll}
\Gamma(\widetilde{\mathcal{T}}) \ni t &=& [u, v] & \mapsto\ \widetilde{t}^{\mathcal{G}} : u \mapsto v \qquad G\cap\widetilde{P}\text{-equivariant} \\
&=& \big[[u,\widetilde{p}], \widetilde{p}^{-1}v\big] & \mapsto\ \widetilde{t}^{\widetilde{\mathcal{G}}} : [u,\widetilde{p}] \mapsto \widetilde{p}^{-1}v \quad \widetilde{P}\text{-equivariant}
\end{array}$$

So with the inclusion $\mathcal{G} \ni u \mapsto [u, id] \in \widetilde{\mathcal{G}}$ we obtain $\widetilde{t}^{\widetilde{\mathcal{G}}}|_{\mathcal{G}} = \widetilde{t}^{\mathcal{G}}$. According to Proposition 3.7 the Cartan connection on the Cartan bundle $\mathcal{G}$ of the CR manifold M induces the tractor connection on the tractor bundle $\widetilde{\mathcal{T}}$ which again induces the Cartan connection on the Cartan bundle $\widetilde{\mathcal{G}}$. Thus we can write for the Cartan connection $\widetilde{\omega}$ of the $\widetilde{P}$-principal bundle $\widetilde{\mathcal{G}}$ and a vector $\overline{X} \in T_u\mathcal{G}$ with projection $X \in T_x\mathcal{F}$ and $t \in \Gamma(\widetilde{\mathcal{T}})$:

$$\begin{aligned}
\widetilde{\omega}([\overline{X},0])\big(\widetilde{t}^{\widetilde{\mathcal{G}}}([u,id])\big) &= \underline{[u,id]}^{-1}\big(\nabla^{\widetilde{\mathcal{T}}}_X t\big)_x - [\overline{X},0]\big(\widetilde{t}^{\widetilde{\mathcal{G}}}\big)_{[u,id]} \\
&= \underline{[u,id]}^{-1}\big(\big[u, \overline{X}(\widetilde{t}^{\mathcal{G}})_u + \omega_u(\overline{X})(\widetilde{t}^{\mathcal{G}})_u\big]\big) - [\overline{X},0]\big(\widetilde{t}^{\widetilde{\mathcal{G}}}\big)_{[u,id]} \\
&= \overline{X}(\widetilde{t}^{\mathcal{G}})_u + \omega_u(\overline{X})(\widetilde{t}^{\mathcal{G}})_u - \underbrace{[\overline{X},0]\big(\widetilde{t}^{\widetilde{\mathcal{G}}}\big)_{[u,id]}}_{=\overline{X}(\widetilde{t}^{\mathcal{G}})_u} \\
&= \omega_u(\overline{X})(\widetilde{t}^{\mathcal{G}})_u.
\end{aligned}$$

Since the Cartan connection has to be $\widetilde{P}$-equivariant and reproduce the generators of the fundamental vector fields this already implies that $\widetilde{\omega}$ is given by

$$\widetilde{\omega}_{[u,\widetilde{p}]} = Ad(\widetilde{p}^{-1}) \circ \pi^*_{\mathcal{G}}\omega + \pi^*_{\widetilde{P}}\omega_{\widetilde{P}},$$

where $\omega_{\widetilde{P}}$ denotes the Maurer Cartan form of $\widetilde{P}$ and $\pi_{\mathcal{G}}$ and $\pi_{\widetilde{P}}$ are the obvious projections.

As we have already seen in the beginning of this proof $(\widetilde{\mathcal{T}}, \widetilde{g}, \nabla^{\widetilde{\mathcal{T}}})$ is a standard tractor bundle for the Fefferman space, it now remains to check the normality condition, $\Omega_-^{\widetilde{\omega}} \overset{!}{=} 0$. We need to take a look at the curvature of the Cartan connection $\widetilde{\omega}$. With similar calculations as in the proof of Lemma 3.8 and keeping in mind that the bracket $[\cdot,\cdot]^\wedge$ for one-forms is symmetric we obtain for the curvature:

$$
\begin{aligned}
\Omega^{\widetilde{\omega}} &= \Omega^{Ad(\pi_{\widetilde{P}}^{-1})\circ\pi_{\mathcal{G}}^*\omega+\pi_{\widetilde{P}}^*\omega_{\widetilde{P}}} \\
&= d\big(Ad(\pi_{\widetilde{P}}^{-1})\circ\pi_{\mathcal{G}}^*\omega+\pi_{\widetilde{P}}^*\omega_{\widetilde{P}}\big) + \tfrac{1}{2}\big[Ad(\pi_{\widetilde{P}}^{-1})\circ\pi_{\mathcal{G}}^*\omega+\pi_{\widetilde{P}}^*\omega_{\widetilde{P}}, Ad(\pi_{\widetilde{P}}^{-1})\circ\pi_{\mathcal{G}}^*\omega+\pi_{\widetilde{P}}^*\omega_{\widetilde{P}}\big]^\wedge \\
&= \Omega^{Ad(\pi_{\widetilde{P}}^{-1})\circ\pi_{\mathcal{G}}^*\omega} + \underbrace{\Omega^{\pi_{\widetilde{P}}^*\omega_{\widetilde{P}}}}_{=\pi_{\widetilde{P}}^*\Omega^{\omega_{\widetilde{P}}}\equiv 0} \\
&\quad + \big[Ad(\pi_{\widetilde{P}}^{-1})\circ\pi_{\mathcal{G}}^*\omega, \pi_{\widetilde{P}}^*\omega_{\widetilde{P}}\big]^\wedge \\
&= d\left(Ad(\pi_{\widetilde{P}}^{-1})\circ\pi_{\mathcal{G}}^*\omega\right) + \tfrac{1}{2}\left[Ad(\pi_{\widetilde{P}}^{-1})\circ\pi_{\mathcal{G}}^*\omega, Ad(\pi_{\widetilde{P}}^{-1})\circ\pi_{\mathcal{G}}^*\omega\right]^\wedge \\
&\quad + \left[Ad(\pi_{\widetilde{P}}^{-1})\circ\pi_{\mathcal{G}}^*\omega, \pi_{\widetilde{P}}^*\omega_{\widetilde{P}}\right]^\wedge \\
&= \underbrace{Ad(\pi_{\widetilde{P}}^{-1})\circ d\left(\pi_{\mathcal{G}}^*\omega\right) - \left[Ad(\pi_{\widetilde{P}}^{-1})\circ\pi_{\mathcal{G}}^*\omega, \pi_{\widetilde{P}}^*\omega_{\widetilde{P}}\right]^\wedge}_{\text{see proof of Lemma 3.8}} + \tfrac{1}{2}Ad(\pi_{\widetilde{P}}^{-1})\circ\left[\pi_{\mathcal{G}}^*\omega, \pi_{\mathcal{G}}^*\omega\right]^\wedge \\
&\quad + \left[Ad(\pi_{\widetilde{P}}^{-1})\circ\pi_{\mathcal{G}}^*\omega, \pi_{\widetilde{P}}^*\omega_{\widetilde{P}}\right]^\wedge \\
&= Ad(\pi_{\widetilde{P}}^{-1})\circ\Omega^{\pi_{\mathcal{G}}^*\omega} \\
&= Ad(\pi_{\widetilde{P}}^{-1})\circ\pi_{\mathcal{G}}^*\Omega^{\omega}.
\end{aligned}
$$

In the conformal case we have $Ad(\widetilde{p}_0)(\widetilde{\mathfrak{g}_{-1}}) = \widetilde{\mathfrak{g}_{-1}}$ and $Ad(\widetilde{p}_0)(\widetilde{\mathfrak{g}}_1) = \widetilde{\mathfrak{g}}_1$ for all elements $\widetilde{p}_0 \in \widetilde{G}_0$ (see for example [Feh05]). Furthermore for elements $\widetilde{p} \in \widetilde{P}$ it holds $Ad(\widetilde{p})(\widetilde{\mathfrak{g}}^i) \subset \widetilde{\mathfrak{g}}^i$. Thus the vanishing of the $\widetilde{\mathfrak{g}}_{-1}$ component of the curvature of the Cartan connection $\widetilde{\omega}$ requires the $\widetilde{\mathfrak{g}}_{-1}$ component of the curvature of the Cartan connection ω to vanish, $pr_{\widetilde{\mathfrak{g}}_{-1}} \circ \Omega^{\omega} \overset{!}{\equiv} 0$. However this means Ω^{ω} has to have values in $\mathfrak{g} \cap \widetilde{\mathfrak{p}} \subset \mathfrak{p} \subset \mathfrak{g}$, that is ω has to be torsion free. This however is implied by the integrability of the CR-structure of M as is discussed in Section 4.5. So we have proven that $(\widetilde{\mathcal{T}}, \widetilde{\mathcal{T}}^1, \widetilde{g}, \nabla^{\widetilde{\mathcal{T}}})$ is a standard tractor bundle of the Fefferman space where $\nabla^{\widetilde{\mathcal{T}}}$ is normal, the corresponding Cartan geometry is given by $\widetilde{\mathcal{G}} = \mathcal{G} \times_{G\cap\widetilde{P}} \widetilde{P}$ and $\widetilde{\omega}_{[u,\widetilde{p}]} = Ad(\widetilde{p}^{-1})\circ\pi_{\mathcal{G}}^*\omega + \pi_{\widetilde{P}}^*\omega_{\widetilde{P}}$.

□

5.6 The S^1-Action

We will now take a closer look at the fundamental vector field generated by the S^1-action on the Fefferman space.

Multiplication by the complex number i gives an endomorphism of $\widetilde{\mathcal{T}}$

$$
\begin{aligned}
i : \widetilde{\mathcal{T}} = \mathcal{G} \times_{G\cap\widetilde{P}} \mathbb{C}^{1,n+1} &\longrightarrow \widetilde{\mathcal{T}} \\
[u,v] &\mapsto [u, iv].
\end{aligned}
$$

Since the subgroup $G\cap\widetilde{P}$ is complex linear i commutes with all elements of it. Furthermore $i = \frac{d}{dt}e^{it}|_{t=0}$ is an element of Lie algebra $\widetilde{\mathfrak{g}} = \mathfrak{o}(2, 2n+1)$ since we have for the inner product $Re\langle iX, Y\rangle_{1,n+1} + Re\langle X, iY\rangle_{1,n+1} = Re\big(-i\langle X,Y\rangle_{1,n+1} + i\langle X,Y\rangle_{1,n+1}\big) = 0$.

Due to i commuting with every $g \in G\cap\widetilde{P}$ the constant map $\mathbb{I} : \mathcal{G} \ni u \mapsto i \in \widetilde{\mathfrak{g}}$ is $G\cap\widetilde{P}$-equivariant, $\mathbb{I}(R_p u) = Ad(p^{-1})i = i$. Thus we obtain a section of the adjoint tractor bundle of the Fefferman space, denoting it with the same symbol,

$$
\begin{aligned}
\mathbb{I} : \mathcal{F} &\longrightarrow \mathcal{G} \times_{G\cap\widetilde{P}} \widetilde{\mathfrak{g}} \simeq \widetilde{\mathcal{G}} \times_{\widetilde{P}} \widetilde{\mathfrak{g}} \\
x &\mapsto [u, i] = [R_p u, i] \text{ with } \pi(u) = x
\end{aligned}
$$

and a vector field

$$\begin{array}{rcl} \widetilde{i} : \mathcal{F} & \longrightarrow & T\mathcal{F} \\ x & \mapsto & d\pi_{\mathcal{F}} \circ \widetilde{\omega}^{-1}(i). \end{array}$$

Proposition 5.2 *The vector field $\widetilde{i} \in \mathfrak{X}(\mathcal{F})$ is nowhere vanishing and generates the vertical bundle of $\mathcal{F} \longrightarrow M$. Furthermore $\widetilde{i}$ is a conformal Killing vector field and the curvature $\Omega^{\widetilde{\omega}}$ vanishes if lifts of $\widetilde{i}$ are inserted.*

Proof: We fix an element $p \in \mathfrak{p}$, which acts by multiplication with the complex number i on the real line $\widetilde{\ell}$. Since p does not preserve the null line $\widetilde{\ell}$ it is no element of $\mathfrak{g} \cap \widetilde{\mathfrak{p}}$. However $(i \cdot id - p)|_{\widetilde{\ell}} \equiv 0$ and so especially $i \cdot id - p \in \widetilde{\mathfrak{p}}$. Thus the isomorphism $\mathfrak{g}/_{\mathfrak{g}\cap\widetilde{\mathfrak{p}}} \longrightarrow \widetilde{\mathfrak{g}}/_{\widetilde{\mathfrak{p}}}$ known from Section 5.2 maps the element $p + \mathfrak{g} \cap \widetilde{\mathfrak{p}} \neq 0$ to $i \cdot id + \widetilde{\mathfrak{p}}$, that is to say $i \cdot id$ is no element of $\widetilde{\mathfrak{p}}$. Considering the projection

$$\begin{array}{ccccc} \pi_{\mathfrak{g}/\mathfrak{p}} : \widetilde{\mathfrak{g}}/_{\widetilde{\mathfrak{p}}} \ni & i + \widetilde{\mathfrak{p}} & \mapsto & 0 & \in \mathfrak{g}/_{\mathfrak{p}} \\ \updownarrow \widetilde{\omega}_x & & & \omega_{\pi(x)} \updownarrow & \\ T_x\mathcal{F} & & & T_{\pi(x)}M & \end{array}$$

we see, that $\widetilde{i} = d\pi_{\mathcal{F}} \circ \widetilde{\omega}^{-1}(i)$ is nowhere vanishing and generates the vertical bundle of $\mathcal{F} \longrightarrow M$.

Another way of describing the vector field $\widetilde{i}$ is obtained by using an element $p \in \mathfrak{p}$, which acts by multiplication with i on $\widetilde{\ell}$. As discussed above $p + \mathfrak{g} \cap \widetilde{\mathfrak{p}}$ is mapped to $i \cdot id + \widetilde{\mathfrak{p}}$ and therefore we get

$$\widetilde{i} = d\pi_{\mathcal{F}} \circ \omega^{-1}(p) = d\pi_{\mathcal{F}} \left(\frac{d}{dt} R_{\phi_t} u \Big|_{t=0} \right) \text{ with } \phi_0 = id \text{ and } \frac{d}{dt}\phi_t|_{t=0} = p.$$

We can set for example

$$p = \begin{pmatrix} i & & \\ & -\frac{2i}{n} I_n & \\ & & i \end{pmatrix} \text{ and } \phi_t = \begin{pmatrix} e^{it} & & \\ & e^{-\frac{2it}{n}} I_n & \\ & & e^{it} \end{pmatrix}.$$

In order to see that $\widetilde{i}$ is a conformal Killing vector field we check the Lie derivative of a metric from the conformal class of the Fefferman space in direction of the vector field $\widetilde{i}$. Recall that the conformal class of $\mathcal{F}$ was given by the conformal class of the Möbius space with the help of the Cartan connection ω. Hence the Lie derivative is

$$\begin{array}{rcl} L_{\widetilde{i}} g & = & \frac{d}{dt}\left(\phi_t^* g \right)\big|_{t=0} \\ & = & \frac{d}{dt}\left(g(\phi_t^*\omega(\cdot) + \mathfrak{g} \cap \widetilde{\mathfrak{p}}, \phi_t^*\omega(\cdot) + \mathfrak{g} \cap \widetilde{\mathfrak{p}})\right)\big|_{t=0} \\ & = & \frac{d}{dt}\left(g(Ad(\phi_t^{-1}) \circ \omega(\cdot) + \mathfrak{g} \cap \widetilde{\mathfrak{p}}, Ad(\phi_t^{-1}) \circ \omega(\cdot) + \mathfrak{g} \cap \widetilde{\mathfrak{p}})\right)\big|_{t=0}. \end{array}$$

However the action $Ad(\phi_t^{-1})$ leaves the inner product invariant and we obtain the result wanted:

$$\begin{array}{rcl} L_{\widetilde{i}} g & = & \frac{d}{dt}\left(g(\omega(\cdot) + \mathfrak{g} \cap \widetilde{\mathfrak{p}}, \omega(\cdot) + \mathfrak{g} \cap \widetilde{\mathfrak{p}})\right)\big|_{t=0} \\ & = & 0. \end{array}$$

Thus $\widetilde{i}$ is a conformal Killing vector field.

Since the curvature of the Cartan connection $\widetilde{\omega}$ is described by the curvature of ω and since lifts $\bar{\widetilde{i}}$ of the vector field $\widetilde{i}$ are vertical with respect to the Cartan bundle $(\mathcal{G}, \pi_M, M; \omega)$ we obviously obtain

$$\Omega^{\widetilde{\omega}}(\bar{\widetilde{i}}, \cdot) = Ad(\pi_{\widetilde{P}}^{-1}) \circ \pi_{\mathcal{G}}^* \Omega^{\omega}(\bar{\widetilde{i}}, \cdot) \equiv 0.$$

$\square$

5.7 The Construction of [BL04] Revisited

Using several of the isomorphisms of Section 5.4 we get similar results for the construction of the Fefferman space according to [BL04]. As we have seen we have

$$\begin{array}{rcl} \underline{\mathcal{G}} \times_{\left[\underline{P}, \widetilde{Ad}=Ad^{\mathbb{C}}_{\mathfrak{g}_2} \wedge \left(Ad^{\mathbb{C}}_{\mathfrak{g}_1}\right)^{\wedge n}\right]} \bigwedge^{n+1,0}(\mathfrak{g}/\mathfrak{p}) & \longrightarrow & \mathcal{K} \\ {[u, X_2 \wedge X_{11} \wedge \cdots \wedge X_{1n}]} & \mapsto & \left(pr_{\mathbb{C}T} \circ d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})_u^{-1}(X_2^*)\right)^* \\ & & \wedge\left(d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})_u^{-1}(X_{11}^*)\right)^* \\ & & \wedge \cdots \wedge \left(d\underline{\pi}^{\mathbb{C}} \circ (\underline{\omega}^{\mathbb{C}})_u^{-1}(X_{1n}^*)\right)^*. \end{array}$$

The $\underline{P}$-action here is according to Subsection 4.2.2 given by

$$\widetilde{Ad}(A)(E') \;=\; \overline{a}^{n+2}E'$$

for $A = \begin{pmatrix} a & -aZ^*B & b \\ 0 & B & Z \\ 0 & 0 & \overline{a}^{-1} \end{pmatrix}$ modulo the center of G. However in our calculations it is sufficient to neglect the center of G since it cancels out anyways. This is equivalent to $\underline{P}$ acting on the complex null line $\ell = \mathbb{C}f_-$ by

$$\begin{array}{rcl} \overline{\varrho}^{-1} : \underline{P} \times \ell & \longrightarrow & \ell \\ (A, X) & \mapsto & \overline{A}^{n+2}X = \overline{a}^{n+2}X. \end{array}$$

I.e. $\mathcal{K} \simeq \underline{\mathcal{G}} \times_{[\underline{P}, \overline{\varrho}^{-1}]} \ell$.
And we obtain

$$\mathcal{F} = (\mathcal{K} \setminus \{0\})/_{\mathbb{R}^+} \simeq \underline{\mathcal{G}} \times_{[\underline{P}, \overline{\varrho}^{-1}]} (\ell \setminus \{0\})/_{\mathbb{R}^+}.$$

Note that the action $\overline{\varrho}^{-1}$ is transitive on ℓ and therefore transitive on the rays $(\ell \setminus \{0\})/_{\mathbb{R}^+}$. Now the stabiliser of the real ray $\mathbb{R}^+ f_- \subset \ell \setminus \{0\}$ with respect to the action $\overline{\varrho}^{-1}$ is $(\widetilde{P}^+ \cap G) \cdot Z(G)$, where $\widetilde{P}^+$ denotes the positiv part of the stabiliser of a real line in $\widetilde{G}$ as before, i.e.

$$\widetilde{P}^+ := \left\{A \in \widetilde{G} \;\middle|\; Af_- = \lambda f_- \text{ for a } \lambda \in \mathbb{R}^+\right\}.$$

So with $(\widetilde{P}^+ \cap G) \cdot Z(G) \subset \underline{P}$ we can write

$$\mathcal{F} \simeq \underline{\mathcal{G}}/_{(\widetilde{P}^+ \cap G) \cdot Z(G)}.$$

The $\widetilde{\mathfrak{g}}_{-1}$-component of the Cartan connection of the Fefferman space is given by the displacement form of the bundle of all conformal reperes with respect to the metric h_θ (see [Feh05] for example). And with $\widetilde{P}$ acting on this component by conformal isomorphisms the conformal class of h_θ is the same as the one defined with the help of the metric of the Möbius space by

$$g_{\text{Möb}}(\omega_{\mathcal{F}}(\overline{X}) + \widetilde{\mathfrak{p}}, \omega_{\mathcal{F}}(\overline{Y}) + \widetilde{\mathfrak{p}}),$$

where $\overline{X}$ and $\overline{Y}$ are lifts of vector fields $X, Y \in \mathfrak{X}(\mathcal{F})$.
This structure we also have for the construction of [CG08]. Here the conformal structure was defined via

$$g(X_x, Y_x) := g_{\text{Möb}}\left((\omega_M)_u(\overline{X}) + \mathfrak{g} \cap \widetilde{\mathfrak{p}}, (\omega_M)_u(\overline{Y}) + \mathfrak{g} \cap \widetilde{\mathfrak{p}}\right) \text{ see Section 5.4.}$$

As above $\widetilde{P}$ acts by conformal isomorphism on the $\widetilde{\mathfrak{g}}_{-1}$-component. And so with the Cartan connection of the Fefferman space being defined as $(\omega_{\mathcal{F}})_{[u,\widetilde{p}]} = Ad(\widetilde{p}^{-1}) \circ \pi^*_{\mathcal{G}_M}\omega_M + \pi^*_{\widetilde{P}}\omega_{\widetilde{P}}$ this is the same as the conformal class defined by

$$g(X_x, Y_x) := g_{\text{Möb}}\left((\omega_{\mathcal{F}})_{[u,\widetilde{p}]}(\overline{X}) + \widetilde{\mathfrak{p}}, (\omega_{\mathcal{F}})_{[u,\widetilde{p}]}(\overline{Y}) + \widetilde{\mathfrak{p}}\right).$$

Thus with the help of the Cartan connections we obtain conformal pointwise isomorphisms

$$T_x\mathcal{F}_{\text{[CG08]}} \longleftrightarrow \widetilde{\mathfrak{g}}_{-1} \longleftrightarrow T_x\mathcal{F}_{\text{[BL04]}}.$$

Joined by the coverings from Section 5.4 we obtain that those coverings are conformal, i.e. the pull back of both conformal structures to $\mathcal{E}(1,0) \times_{[S^1,\lambda^{-1}]} S^1$ gives the same conformal structure.

$$\begin{tikzcd} & \mathcal{E}(1,0) \times_{[S^1,\lambda^{-1}]} S^1 \arrow[dl, "2:1"'] \arrow[dr, "(n+2):1"] & \\ \mathcal{F}_{\text{[CG08]}} & & \mathcal{F}_{\text{[BL04]}} \end{tikzcd}$$

Especially local results on the Fefferman space can be transfered from one construction to the other.

Chapter 6

The Cartan Boundary

While it is quite easy to define a boundary of a Riemannian manifold, physicists encountered many challenges when searching for ways to define the boundary of a space-time. One of the first approaches using geodesics soon turned out to be dissatisfactory. Geroch for example constructed a space-time which is geodesically complete but still contains an inextendable curve of bounded acceleration. I.e. in such a space-time a rocket charged with a finite amount of fuel would actually reach "the boundary" of this universe. The idea of a boundary constructed with the help of bundles was suggested by Ehresman in 1957 and reformulated and studied by Schmidt in [Sch71], [Sch73] and [Sch74] for semi-Riemannian and conformal manifolds. This boundary of space-times and conformal manifolds was at first denoted with ∂_b due to the underlying bundle construction. However since the bundle in question is the Cartan bundle and since this construction is applicable for all Cartan geometries we will denote this boundary by ∂_{CB}. We hope that in doing so, we will avoid a mix up with other boundaries such as the conformal or the causal boudary.
This boundary construction is of great interest not only because of the broad usability but also since it is intrinsic and enlarges other intrinsic boundary definitions such as geodesical boundaries or ba-boundaries (bounded acceleration) as we will see later. In the case of Riemannian manifolds the Cartan boundary actually coincides with the metric boundary. Thus in this sense the Cartan boundary expands the definition of the metric boundary to a broad variety of manifolds, namely the Cartan geometries. In this chapter we will define the Cartan boundary and take a look at several examples. Properties such as completeness, relations to other boundary definitions and Hausdorffness will be studied.

6.1 Definition of the Cartan Boundary

Here we will give a general definition of the Cartan boundary of a Cartan geometry.
Let $\pi : \mathcal{G} \longrightarrow M$ be a Cartan geometry of type (G, P) and $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ its Cartan connection. Without loss of generality we assume $\mathcal{G}$ to be connected. Otherwise the Cartan boundary can be constructed for each connected component seperately.
Let $(a_1, \ldots, a_r)$ be a basis of $\mathfrak{g}$. Hence the ω-constant vector fields $A_i(u) := (\omega_u)^{-1}(a_i)$, $i = 1, \ldots, r$ form a global frame of $\mathcal{G}$. Using this we can define a Riemannian metric ϱ on $\mathcal{G}$ via $\varrho(A_i, A_j) := \delta_{ij}$. We obtain the corresponding distance d_ϱ which makes $(\mathcal{G}, d_\varrho)$ a metric space:

$$\begin{array}{rcl} d_\varrho : \mathcal{G} \times \mathcal{G} & \longrightarrow & \mathbb{R}_+ \\ (u, v) & \mapsto & d_\varrho(u, v) := \inf \{\ell(\gamma) \mid \gamma \in \Omega(u, v)\}. \end{array}$$

Here $\Omega(u, v)$ denotes the space of all piecewise smooth paths from u to v, and the length ℓ of a curve $\gamma : [0, 1] \longrightarrow \mathcal{G}$ is defined as

$$\ell(\gamma) := \int_0^1 \sqrt{\varrho(\gamma'(t), \gamma'(t))}dt.$$

By fixing an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ such that $(a_1, \ldots, a_r)$ is an orthonormal basis and $\| \cdot \|$ denoting the corresponding norm on $\mathfrak{g}$ we can also write for the length of the curve γ

$$\ell(\gamma) = \int_0^1 \|\omega(\dot{\gamma})\| dt.$$

We define $\overline{\mathcal{G}}$ to be the Cauchy completion of $(\mathcal{G}, d_\varrho)$ in the usual way. I.e. for the metric space $(\mathcal{G}, d_\varrho)$ we denote with $CF(\mathcal{G}, d_\varrho)$ the space of all Cauchy sequences. Two Cauchy sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in CF(\mathcal{G}, d_\varrho)$ are defined to be equivalent, $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$, if the limit $\lim_{n\to\infty} d_\varrho(x_n, y_n)$ is zero. Now we set

$$\overline{\mathcal{G}} := CF(\mathcal{G}, d_\varrho)/_\sim.$$

The metric space $\mathcal{G}$ can be considered as a subset of $\overline{\mathcal{G}}$ by

$$\begin{array}{rcl} \mathcal{G} & \longrightarrow & \overline{\mathcal{G}} \\ x & \mapsto & [(x_n = x)], \end{array}$$

and the distance d_ϱ can be prolonged to $\overline{\mathcal{G}}$ via

$$d_\varrho\Big(\big[(x_n)_{n\in\mathbb{N}}\big], \big[(y_n)_{n\in\mathbb{N}}\big]\Big) := \lim_{n\to\infty} d_\varrho(x_n, y_n).$$

Then $(\overline{\mathcal{G}}, d_\varrho)$ is a complete metric space and $\mathcal{G} \subset \overline{\mathcal{G}}$ is dense.
Choosing in the beginning a different basis $(a'_1, \ldots, a'_r)$ of $\mathfrak{g}$ gives a metric $d_{\varrho'}$ which is equivalent to d_ϱ. Then $\overline{\mathcal{G}}' = CF(\mathcal{G}, d_{\varrho'})$ is identical to $\overline{\mathcal{G}}$ and the metrics d_ϱ and $d_{\varrho'}$ are equivalent on the Cauchy completion $\overline{\mathcal{G}}$.
Thus the construction of the topological space $\overline{\mathcal{G}}$ is independent of the basis $(a_1, \ldots, a_r)$ chosen.
By definition the boundary points of $\mathcal{G}$ are given by the non-converging Cauchy sequences. However when searching for boundary points it is also possible to look for piecewise smooth curves $\gamma : [0,1) \longrightarrow \mathcal{G}$ which are inextendable in $\mathcal{G}$ and of finite length.
Any such curve $\gamma : [0,1) \longrightarrow \mathcal{G}$ defines a boundary point since for a sequence $t_n \overset{n\to\infty}{\longrightarrow} 1$ the corresponding sequence of the points of the curve $(\gamma(t_n))_{n=1}^\infty$ is a non-converging Cauchy sequence. The other way round given a non-converging Cauchy sequence $(x_n)_{n=1}^\infty$ and fixing some $0 < q < 1$ we can find an increasing sequence of natural numbers n_k such that for all $k \in \mathbb{N}$ and all $n, m \geq n_k$ we have $d_\rho(x_n, x_m) < q^k$. According to the definition of the distance d_ϱ we have a piecewise smooth curve $\gamma_k : \left[1 - \frac{1}{k}, 1 - \frac{1}{k+1}\right] \longrightarrow \mathcal{G}$ connecting x_{n_k} and $x_{n_{k+1}}$ with length $\ell(\gamma_k) < 2q^k$, i.e. $\gamma_k(1 - \frac{1}{k}) = x_{n_k}$ and $\gamma_k(1 - \frac{1}{k+1}) = x_{n_{k+1}}$. So for the piecewise smooth curve $\gamma : [0,1) \longrightarrow \mathcal{G}$ defined by $\gamma|_{[1-\frac{1}{k}, 1-\frac{1}{k+1}]} := \gamma_k$ we have $\ell(\gamma) \leq \sum_{k=1}^\infty \ell(\gamma_k) < 2\sum_{k=1}^\infty q^k = \frac{2q}{1-q}$. Hence γ is a piesewise smooth curve of finite length whose image contains the subsequence (x_{n_k}) and therefore γ is inextendable, of finite length and defines the same boundary point as the Cauchy sequence (x_n).
Please note that two inextensible curves of finite length might define the same boundary point. Here equivalence can be checked by going back to the setting of Cauchy sequences.
We will now prove that for all $p \in P$ the right action $R_p : \mathcal{G} \longrightarrow \mathcal{G}$ of p on $\mathcal{G}$ is uniformly continuous with respect to the distance d_ϱ defined by ϱ. If that is proven the right action of P can be uniquely prolonged to $\overline{\mathcal{G}}$.

Lemma 6.1 *For all $p \in P$ the right action $R_p : \mathcal{G} \longrightarrow \mathcal{G}$ of p on $\mathcal{G}$ is uniformly continuous with respect to the distance d_ϱ.*

Proof: Fixing an element $p \in P$ we can write for the length of a curve $\gamma : [0,1] \longrightarrow \mathcal{G}$

$$\begin{array}{rcl} \ell(R_p \circ \gamma) & = & \int_0^1 \left\|\omega\left(\frac{d}{dt}(R_p \circ \gamma)\right)\right\| dt \\ & = & \int_0^1 \left\|Ad(p^{-1}) \circ \omega(\dot\gamma)\right\| dt \\ & \leq & \int_0^1 \|Ad(p^{-1})\| \cdot \|\omega(\dot\gamma)\| dt \\ & = & \|Ad(p^{-1})\| \cdot \ell(\gamma). \end{array}$$

Hence for all points $u, v \in \mathcal{G}$ we have $d_\varrho(R_p u, R_p v) \le \|Ad(p^{-1})\| d_\varrho(u, v)$. Thus for all $p \in P$ the right action $R_p : \mathcal{G} \longrightarrow \mathcal{G}$ of p on $\mathcal{G}$ is Lipschitz continuous and therefore uniformly continuous with respect to the distance d_ϱ.

$\square$

Remark 6.1 *As we have seen in the proof above, $\|Ad(p^{-1})\|$ is a Lipschitz constant for $R_p : \mathcal{G} \longrightarrow \mathcal{G}$ and it depends continuously on p. So for some $\delta > 0$ the supremum of all Lipschitz constants $sup_{A \in B_\delta(0)} \|Ad(\exp(-A))\|$ is finite.*

Now the right action of P can be uniquely prolonged to $\overline{\mathcal{G}}$. For $u \in \overline{\mathcal{G}}$ being represented by the Cauchy sequence (u_n) we set $R_p u$ to be the class of the Cauchy sequence $(R_p u_n)$. This is well-defined since we have for two equivalent Cauchy sequences $(x_n)_{n \in \mathbb{N}} \sim (y_n)_{n \in \mathbb{N}}$ that

$$\begin{aligned} \lim\nolimits_{n \to \infty} d_\varrho\big(R_p x_n, R_p y_n\big) &\le \lim\nolimits_{n \to \infty} \|Ad(p^{-1})\| d_\varrho\big(x_n, y_n\big) \\ &= \|Ad(p^{-1})\| \underbrace{\lim_{n \to \infty} d_\varrho\big(x_n, y_n\big)}_{=0} \\ &= 0. \end{aligned}$$

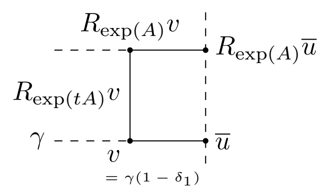
I.e. $(R_p x_n)_{n \in \mathbb{N}}$ and $(R_p y_n)_{n \in \mathbb{N}}$ define the same point in $\overline{\mathcal{G}}$.

Lemma 6.2 *The right action of P is continuous on $\overline{\mathcal{G}}$. More precisely*

$$\begin{aligned} \overline{\mathcal{G}} \times P &\longrightarrow \overline{\mathcal{G}} \\ (\overline{u}, p) &\mapsto R_p \overline{u} \end{aligned}$$

is continuous.

Proof: On P we have the distance d_ϱ in the same way as on $\mathcal{G}$ by defining the left invariant vector fields given by the fixed frame $(a_1, \dots, a_r)$ (or rather this part of this $\mathfrak{g}$-frame, which gives a frame of $\mathfrak{p}$) to be orthonormal. First of all $R_p : \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{G}}$ is Lipschitz continuous with Lipschitz constant $\|Ad(p^{-1})\|$ as we have just seen. Further for any $\overline{u} \in \overline{\mathcal{G}}$ the map $P \ni p \mapsto R_p \overline{u} \in \overline{\mathcal{G}}$ is continuous in the neutral element $e \in P$. Of course this is true for all $\overline{u} \in \mathcal{G}$ so to verify this it is sufficient to find for every $\varepsilon > 0$ and every $\overline{u} \in \overline{\mathcal{G}} \setminus \mathcal{G}$ a $\delta > 0$ such that $d_\varrho(R_{\exp(A)} \overline{u}, \overline{u}) < \varepsilon$ for all $A \in \mathfrak{p}$ with $\|A\| < \delta$. The following picture illustrates the idea of the proof.



Since the exponential map $\exp : \mathfrak{p} \longrightarrow P$ is a local diffeomorphism around zero, we can fix a $\widetilde{\delta} > 0$ such that the restriction $\exp : B_{\widetilde{\delta}}(0) \longrightarrow \exp(B_{\widetilde{\delta}}(0))$ is a diffeomorphism and we set $C_\varepsilon := \sup_{A \in B_{\widetilde{\delta}}(0)} \|Ad(\exp(-A))\| < \infty$, where $\|Ad(\exp(-A))\|$ is the Lipschitz constant of $R_{\exp(A)}$. Hence for all curves γ and all $A \in \mathfrak{p}$ with $\|A\| < \widetilde{\delta}$ we have $\ell(R_{\exp(A)} \circ \gamma) \le C_\varepsilon \ell(\gamma)$. Now let $\gamma : [0, 1) \longrightarrow \mathcal{G}$ be a curve defining $\overline{u}$ and $\overline{\gamma}$ denotes its extension in $\overline{\mathcal{G}}$, that is γ is of finite length and $\overline{\gamma}(1) := lim_{t \to 1} \gamma(t) = \overline{u}$. Choose $\delta_1 > 0$ such that $\ell(\overline{\gamma}|_{[1-\delta_1, 1]}) < \frac{\varepsilon}{2(1 + C_\varepsilon)}$. We set $v := \gamma(1 - \delta_1)$. The length of the curve through v defined by $A \in \mathfrak{p}$ is given by

$$\begin{aligned} \ell\left(R_{\exp(tA)} v|_{[0,1]}\right) &= \int_0^1 \left\|\omega(\widetilde{A})\right\| dt \\ &= \int_0^1 \|A\| \, dt \\ &= \|A\|. \end{aligned}$$

Hence for all $A \in \mathfrak{p}$ with $\|A\| < min\left(\frac{\varepsilon}{2}, \widetilde{\delta}\right) := \delta$ we have

$$\begin{aligned} d_\varrho(\overline{u}, R_{\exp(A)}\overline{u}) &\leq \ell\left(\overline{\gamma}|_{[1-\delta_1,1]}\right) + \ell\left(R_{\exp(tA)}v|_{[0,1]}\right) + \ell\left(R_{\exp(A)} \circ \overline{\gamma}|_{[1-\delta_1,1]}\right) \\ &< \tfrac{\varepsilon}{2(1+C_\varepsilon)} + \|A\| + C_\varepsilon \ell\left(\overline{\gamma}|_{[1-\delta_1,1]}\right) \\ &< \tfrac{\varepsilon}{2(1+C_\varepsilon)} + \tfrac{\varepsilon}{2} + C_\varepsilon \tfrac{\varepsilon}{2(1+C_\varepsilon)} \\ &= \varepsilon. \end{aligned}$$

Now let $[(x_n)_{n\in\mathbb{N}}]$ and $[(y_n)_{n\in\mathbb{N}}]$ be two points in $\overline{\mathcal{G}}$ and $p, \hat{p} \in P$ with $d_\varrho(p,\hat{p})$ sufficiently small, that we have a $v \in \mathfrak{p}$ with $\hat{p} = \exp(v) \cdot p$. Then we have

$$\begin{aligned} & d_\varrho\big(R_p[(x_n)_{n\in\mathbb{N}}], R_{\hat{p}}[(y_n)_{n\in\mathbb{N}}]\big) \\ \leq\ & d_\varrho\big(R_p[(x_n)_{n\in\mathbb{N}}], R_p[(y_n)_{n\in\mathbb{N}}]\big) + d_\varrho\big(R_p[(y_n)_{n\in\mathbb{N}}], R_{\hat{p}}[(y_n)_{n\in\mathbb{N}}]\big) \\ \leq\ & \|Ad(p^{-1})\| d_\varrho\big([(x_n)_{n\in\mathbb{N}}], [(y_n)_{n\in\mathbb{N}}]\big) + d_\varrho\big(R_p[(y_n)_{n\in\mathbb{N}}], R_p \circ R_{\exp v}[(y_n)_{n\in\mathbb{N}}]\big) \\ \leq\ & \|Ad(p^{-1})\| d_\varrho\big([(x_n)_{n\in\mathbb{N}}], [(y_n)_{n\in\mathbb{N}}]\big) + \|Ad(p^{-1})\| d_\varrho\big([(y_n)_{n\in\mathbb{N}}], R_{\exp v}[(y_n)_{n\in\mathbb{N}}]\big). \end{aligned}$$

With $P \ni p \mapsto R_p\overline{u} \in \overline{\mathcal{G}}$ being continuous in the neutral element $e \in P$ for any $\overline{u} \in \overline{\mathcal{G}}$ this can be made arbitrarily small by choosing

$$d\Big(\big([(x_n)_{n\in\mathbb{N}}], p\big), \big([(y_n)_{n\in\mathbb{N}}], \hat{p}\big)\Big) = d_\varrho\big([(x_n)_{n\in\mathbb{N}}], [(y_n)_{n\in\mathbb{N}}]\big) + d_\varrho(p,\hat{p})$$

sufficiently small.
So the right action $\overline{\mathcal{G}} \times P \longrightarrow \overline{G}$ is continuous.

□

We obtain the following diagram

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \overline{\mathcal{G}} \\ \pi_1 \downarrow & & \downarrow \overline{\pi_1} \\ \mathcal{G}/_P = M & \longrightarrow & \overline{M} := \overline{\mathcal{G}}/_P. \end{array}$$

Devision by P gives $\overline{M} := \overline{\mathcal{G}}/_P$. The Cartan boundary of the Cartan geometry $(\mathcal{G}, \pi, M; \omega)$ is now defined as $\partial_{CB}M := \overline{M} \setminus M$.
On $\overline{M}$ and $\partial_{CB}M$ we use the quotient topology, that is to say we require the projection map to be continuous. A subset $U \subset \overline{M}$ is defined to be open if its preimage $\overline{\pi_1}^{-1}(U) \subset \overline{\mathcal{G}}$ is open. Given an open subset $\mathcal{U} \subset \overline{\mathcal{G}}$ the projection $\overline{\pi_1}(\mathcal{U}) \subset \overline{M}$ is also open since $\overline{\pi_1}^{-1} \circ \overline{\pi_1}(\mathcal{U}) = \mathcal{U} \cdot P$ and the right action of P is continuous and open. Hence the projection $\overline{\pi_1} : \overline{\mathcal{G}} \longrightarrow \overline{M}$ is continuous and open.
We have seen, that the boundary points of a Cartan geometry are defined by inextensible curves of finite length. However we want to point out, that vertical curves never define boundary points since any vertical curve $\gamma : [0,1) \longrightarrow \mathcal{G}$ can be written as $\gamma(t) = R_{p(t)}u$ for some $u \in \mathcal{G}$ and a curve $p : [0,1) \longrightarrow P$. The length of γ is the same as the length of p:

$$\begin{aligned} \ell(\gamma) &= \int_0^1 \left\|\omega\big(\tfrac{d}{dt} R_{p(t)}u\big)\right\| dt \\ &= \int_0^1 \left\|\omega\left(dL_{p(t)^{-1}}\dot{p}(t)(R_{p(t)}u)\right)\right\| dt \\ &= \int_0^1 \left\|dL_{p(t)^{-1}}\dot{p}(t)\right\| dt \\ &= \int_0^1 \left\|\omega_P(\dot{p}(t))\right\| dt \\ &= \ell(p). \end{aligned}$$

And since P is a homogeneous space with a left invariant Riemannian metric it is complete. So any vertical curve γ in $\mathcal{G}$ which is of finite length, is given by a curve p in P which is of finite length and therefore extendable. Hence γ itself would be extendable.
Thus there are no inxetensible vertical curves of finite length.

Still the following questions arise:

- Are all boundary points already defined by a special set of inextendable curves of finite length? In Riemannian geometry for example it is sufficiert to consider just the geodesics. Are there curves like that in Cartan geometry?

  For principal bundle connections we can define for example horizontal curves. Some Cartan connections allow such a definition as well. So are in thoses cases the boundary points already given by horizontal curves?

  For all Cartan geometries we have ω-constant curves, whose vector of velocity is mapped by the Cartan connection onto a constant element of the Lie algebra. Is it sufficient to take a look at those curves in order to determine the Cartan boundary?

- Having given a Cartan geometry $(\mathcal{G}, \pi, M; \omega)$ is the Cartan boundary of an open subset $U \subset M$ the same as the topological boundary defined by the embedding $U \hookrightarrow M$?

- If $\gamma : I \longrightarrow \mathcal{G}$ is a curve defining a boundary point, what do we know about $\pi \circ \gamma$?

- What does the defined distance on $\mathcal{G}$ has to do with the distance on M, if we have actually given a Riemannian manifold?

- Do all inextendable geodesics on a semi-Riemannian manifold define boundary points? What does finite length mean here?

6.2 Examples

6.2.1 The Cartan Boundary of the Homogeneous Model

We want to determine the Cartan Boundary of the model space as it is done in [Fra08]. Let (G, P) be a Cartan model, that is to say G is a Lie group, $P \subset G$ a closed subgroup and $G/_P$ connected. The Cartan geometry of the homogeneous space $G/_P$ is given by the principal bundle $(G, \pi, G/_P)$ with structure group P, together with the Maurer Cartan form ω^G as the Cartan connection. Since the Maurer Cartan form is left invariant, the global frame $\big(A_i(u) := (\omega^G_u)^{-1}(a_i)\big)$ is invariant under the left action of G and so is the Riemannian metric ρ.
I.e. we have given a homogeneous space with a left invariant Riemannian metric. Hence the space (G, ρ) is complete and we have for the Cartan boundary

$$\begin{aligned} \partial_{CB} G &= \emptyset \\ \text{and} \quad \partial_{CB} G/_P &= \emptyset. \end{aligned}$$

6.2.2 The Cartan Boundary of the Conformal Space $\mathbb{R}^{p,q}$

The example of space $\mathbb{R}^n$ endowed with the standard conformally flat structure is discussed in [Fra08] as well. The model space for $\mathbb{R}^n$ is the sphere S^n which can be seen as the homogeneous space $O_c(1, n+1)/_{\widetilde{P}}$ where $\widetilde{P}$ is a parabolic subgroup. Details can be found in [Feh05]. Note that for conformal manifolds of dimension $n \geq 3$ a unique Cartan connection, namely the normal Cartan connection, can be distinguished among all Cartan connections. So for conformal manifolds of dimension $n \geq 3$ the Cartan boundary is also uniquely defined by the conformal structure. Hence we consider conformal manifolds of dimension $n \geq 3$.
Using the stereographic projection the space $\mathbb{R}^n$ can be conformally embedded into the sphere S^n. The image is open and missing just one point of the sphere. Hence the Cartan

bundle of $\mathbb{R}^n$ is identified with $O_c(1,n{+}1)$ missing one $\widetilde{P}$-Orbit. Since $O_c(1,n{+}1)$ is complete and the Cartan bundle of $\mathbb{R}^n$ an open and dense subset of it, the Cauchy completion of the Cartan bundle of $\mathbb{R}^n$ is $O_c(1,n+1)$ and the Cartan boundary of $\mathbb{R}^n$ contains just one point.

$$\partial_{CB}\big(\mathbb{R}^n,[\langle\cdot,\cdot\rangle_n]\big)=\{point\}$$

Let us generalise the ideas from above by considering any signature. We start with the standard conformally flat space $\mathbb{R}^{p,q}$ of signature (p,q), $p+q\geq 3$. In this case the model space is the Möbius space $(Q^{p,q},c_{p,q})$, i.e. the projectivated light cone, of the same signature,

$$Q^{p,q}=\mathbb{P}C=\mathbb{P}\{x\in\mathbb{R}^{p+1,q+1}\mid\langle x,x\rangle_{p+1,q+1}=0\}.$$

Details on this construction and the conformal embedding

$$\begin{array}{rcl} i:\mathbb{R}^{p,q} & \longrightarrow & Q^{p,q} \\ x & \mapsto & \mathbb{R}\left(-2\langle x,x\rangle_{p,q}f_0+2x+f_{n+1}\right) \end{array}$$

with $f_0:=\frac{1}{\sqrt{2}}(e_{n+1}-e_0)$ and $f_{n+1}:=\frac{1}{\sqrt{2}}(e_{n+1}+e_0)$ can be found in [Feh05]. The subset $i(\mathbb{R}^{p,q})=\{x\in C\mid\langle x,f_0\rangle_{p+1,q+1}\neq 0\}\subset Q^{p,q}$ is open and dense and $i^*c_{p,q}=[\langle\cdot,\cdot\rangle_{p,q}]$. The Möbius space is a homogeneous space, namely $Q^{p,q}=O_c(p+1,q+1)/_{\widetilde{P}}$, where $\widetilde{P}$ is the stabiliser of the null line $\mathbb{R}f_0$ in $\mathbb{R}^{p+1,q+1}$. Hence the Cartan boundary of the Möbius space is empty. The Cartan bundle of the Möbius space is $\big(O_c(p+1,q+1),\pi,Q^{p,q};\widetilde{P}\big)$ and $O_c(p{+}1,q{+}1)$ is complete. With the help of the conformal embedding i the Cartan bundle of $\mathbb{R}^{p,q}$ can be regarded as a subbundle of $O_c(p+1,q+1)$ and its Cauchy completion is the whole space $O_c(p{+}1,q{+}1)$ since the subset $i(\mathbb{R}^{p,q})\subset Q^{p,q}$ is open and dense. Consequently the Cartan boundary of $\mathbb{R}^{p,q}$ is $Q^{p,q}\setminus i(\mathbb{R}^{p,q})$.

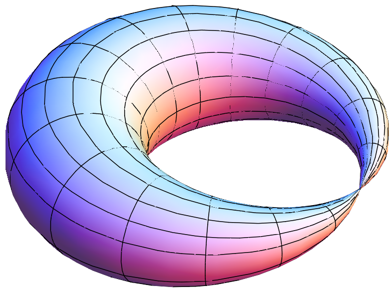
We have

$$\begin{array}{rcl} \partial_{CB}\big(\mathbb{R}^{p,q},[\langle\cdot,\cdot\rangle_{p,q}]\big) & = & Q^{p,q}\setminus i(\mathbb{R}^{p,q}) \\ & = & \mathbb{P}\big\{x\in\mathbb{R}^{p+1,q+1}\mid\langle x,x\rangle_{p+1,q+1}=\langle x,f_0\rangle_{p+1,q+1}=0\big\} \\ & = & \mathbb{P}\big\{x+\lambda f_0\mid x\in\mathbb{R}^{p,q},\langle x,x\rangle_{p,q}=0,\lambda\in\mathbb{R}\big\} \\ & = & \big\{\mathbb{R}(x+\lambda f_0)\mid x\in\mathbb{R}^{p,q},\langle x,x\rangle_{p,q}=0,\lambda\in\mathbb{R}\big\}. \end{array}$$

For $p=0$ we obtain the result from above: $Q^{0,q}\setminus i(\mathbb{R}^{0,q})=\big\{\mathbb{R}f_0\big\}$ is just a point. However for $p,q\geq 1$ we get

$$\begin{array}{rcl} \partial_{CB}\big(\mathbb{R}^{p,q},[\langle\cdot,\cdot\rangle_{p,q}]\big) & = & \big\{\mathbb{R}(\mu x+\lambda f_0)\mid x\in Q^{p-1,q-1},\mu,\lambda\in\mathbb{R}\big\} \\ & = & \big\{\mathbb{R}(\mu x+\lambda f_0)\mid x\in Q^{p-1,q-1},\mu,\lambda\in\mathbb{R}\text{ with }\mu\geq 0\text{ and }\mu^2+\lambda^2=1\big\} \\ & = & Q^{p-1,q-1}\times S^1_+\Big/\sim \\ & & \text{with } S^1_+:=\{e^{i\varphi}\mid 0\leq\varphi\leq\pi\} \\ & & \text{and } (\mathbb{R}x,e^{i\varphi})\sim(\mathbb{R}y,e^{i\psi})\text{ if }\varphi,\psi\in\{0,\pi\}. \end{array}$$

The conformal class $c_{p,q}$ is degenerated along S^1 and the restriction of $c_{p,q}$ to any subset $Q^{p-1,q-1} \times \{e^{i\varphi}\}$ for $\varphi \in (0,\pi)$ is exactly the conformal class $c_{p-1,q-1}$ of $Q^{p-1,q-1}$.
For example the Cartan boundary of $\mathbb{R}^{1,2}$ looks like this:



6.2.3 The Cartan Boundary of Cartan Geometries Modeled on a Reductive Space

Let $G/_P$ be a reductive space, that is the Lie algebra $\mathfrak{g}$ of the Lie group G can be written as a direct sum $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{m}$ such that $\mathfrak{m}$ is $Ad(P)$-invariant, $Ad(P)\mathfrak{m} \subset \mathfrak{m}$. Let furthermore $(\mathcal{G}, \pi, M; \omega)$ be a Cartan geometry of type (G, P). Then the Cartan connection splits into two components $\omega = \underbrace{pr_{\mathfrak{p}} \circ \omega}_{=:\omega_{\mathfrak{p}}} + \underbrace{pr_{\mathfrak{m}} \circ \omega}_{=:\omega_{\mathfrak{m}}}$. The $\mathfrak{p}$-component $\omega_{\mathfrak{p}}$ is a principal bundle connection on the P-principal bundle, that is to say $R_p^*\omega_{\mathfrak{p}} = Ad(p^{-1}) \circ \omega_{\mathfrak{p}}$ for all $p \in P$ and $\omega_{\mathfrak{p}}(\widetilde{X}) = X$ for all $X \in \mathfrak{p}$. The $\mathfrak{m}$-component $\omega_{\mathfrak{m}}$ of the Cartan connection is P-equivariant as well, $R_p^*\omega_{\mathfrak{m}} = Ad(p^{-1}) \circ \omega_{\mathfrak{m}}$ for all $p \in P$, and furthermore its kernel is the vertical tangent space, $ker(\omega_{\mathfrak{m}})_u = Tv_u\mathcal{G}$. Hence $\omega_{\mathfrak{m}}$ is a soldering form also named displacement form.
The other way round a soldering form $\theta \in \Omega^1(\mathcal{G}, \mathfrak{m})$ combined with a P-principal bundle connection $A \in \Omega^1(\mathcal{G}, \mathfrak{p})$ gives a Cartan connection $\omega = \theta + A \in \Omega^1(\mathcal{G}, \mathfrak{g})$. We will therefore write for every Cartan connection of a Cartan geometry modeled on a reductive space $\omega = \theta + A$, where θ is a soldering form and A a principal bundle connection.
A curve $\gamma : I \longrightarrow \mathcal{G}$ is said to be horizontal if $A(\dot{\gamma})$ vanishes. For any curve $\gamma : [0,1) \longrightarrow \mathcal{G}$ we can find a curve $p : [0,1) \longrightarrow P$ with $p(0) = e$ such that $R_p \circ \gamma$ is horizontal, since the equation

$$\begin{aligned} 0 &\overset{!}{=} A\left(\tfrac{d}{dt}(R_p \circ \gamma)\right) \\ &= A\left(dR_p \circ \dot{\gamma} + \widetilde{dL_{p^{-1}}\dot{p}}\right) \\ &= Ad(p^{-1}) \circ \underbrace{A(\dot{\gamma})}_{\in \mathfrak{p}} + dL_{p^{-1}}\dot{p} \end{aligned}$$

is solvable in P. We denote the horizontal curve $R_p \circ \gamma$ by γ^*.
In the semi-Riemannian case this has a nice interpretation. For a semi-Riemannian manifold (M, g) of signature $sign(g) = (p, q)$ the Cartan bundle is given as the bundle of all orthonormal reperes $\mathcal{O}(M, g)$ and the $O_c(p, q)$-principal bundle connection is the Levi-Civita connection A^g which is joined by the displacement form θ to create the Cartan connection, $\omega = A^g + \theta$. Then the horizontal curves γ^* are parallely propagated frames along the projected curves $\gamma = \pi \circ \gamma^*$. Consequently the length of a horizontal curve γ^* is the

length of its projection γ messured in the parallely propagated frame γ^*. This is exactly what physicists call the affine length of the curve γ (see for example [HE73]). To determine the affine length of a curve $\gamma : I \longrightarrow M$ a frame is parallely propagated along the curve, $(e_i) : I \longrightarrow O_c(p,q)$ and $\pi \circ (e_i) = \gamma$. The coordinates of the vector of velocity with respect to the parallely propagated frame are determined, $\dot{\gamma} = (v_1, \ldots, v_n)^t \cdot (e_1, \ldots, e_n)$. The euklidian length of the coordinates is computed and finally integrated over the whole intervall, $\ell_{affin}(\gamma) = \int_I \|(v_1, \ldots, v_n)\| dt$.
Note that horizontal curves are not necessarily shorter than other curves with the same projection and the same starting point. This can already be seen in the flat two dimensional Lorentzian case, $\mathbb{R}^{1,1}$ where the Cartan connection is given by the Levi-Civita connection plus the displacementform. The length of the horizontal curve $\gamma^* : [0,1] \longrightarrow \mathcal{O}(1,1)$ defined as $\gamma^*(t) = (e_1, e_2)_{(xt,xt)}$ for some fixed number $x \in \mathbb{R}_+$ is

$$\begin{aligned} \ell(\gamma^*) &= \int_0^1 \|\theta(\dot{\gamma}^*(t))\| dt \\ &= \int_0^1 \|[(e_1,e_2)]^{-1}(\dot{\gamma}(t))\| dt \\ &= \int_0^1 \sqrt{2} x dt \\ &= \sqrt{2} x. \end{aligned}$$

We set $A(t\alpha) := \begin{pmatrix} \cosh(t\alpha) & \sinh(t\alpha) \\ \sinh(t\alpha) & \cosh(t\alpha) \end{pmatrix}$ for some $\alpha > 0$. The curve $R_{A(t\alpha)} \circ \gamma^*$ has the same projection as γ^* and the same starting point $R_A(0) \circ \gamma^*(0) = \gamma^*(0)$. Its length is

$$\begin{aligned} &\ell(R_{A(t\alpha)} \circ \gamma^*) \\ &= \int_0^1 \left\|(\theta + A^{LC})\left(\tfrac{d}{dt} R_{A(t\alpha)} \circ \gamma^*(t)\right)\right\| dt \\ &= \int_0^1 \left\|(\theta + A^{LC})\left(\tfrac{d}{dt} R_{A(t\alpha)} \circ \gamma^*(t)\right)\right\| dt \\ &= \int_0^1 \left(\left\|\theta\left(\tfrac{d}{dt} R_{A(t\alpha)} \circ \dot{\gamma}^*(t)\right)\right\| + \left\|A^{LC}\left(dR_{A(t\alpha)} \circ \gamma^*(t) + dL_{A(t\alpha)^{-1}} \dot{\widetilde{A(t\alpha)}}\right)\right\|\right) dt \\ &= \int_0^1 \left(\left\|A(t\alpha)^{-1} \begin{pmatrix} x \\ x \end{pmatrix}\right\| + \left\|Ad\big(A(t\alpha)^{-1}\big) \circ \underbrace{A^{LC}(\dot{\gamma}^*(t))}_{=0\ (\gamma^* \text{ horizontal})} + dL_{A(t\alpha)^{-1}} \dot{A(t\alpha)}\right\|\right) dt \\ &= \int_0^1 \left(\left\|\begin{pmatrix} \cosh(t\alpha) & -\sinh(t\alpha) \\ -\sinh(t\alpha) & \cosh(t\alpha) \end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix}\right\| + \left\|\alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\|\right) dt \\ &= \int_0^1 \left((\cosh(t\alpha) - \sinh(t\alpha))\sqrt{2}x + \alpha\right) dt \\ &= \int_0^1 \left(\sqrt{2} x e^{-t\alpha} + \alpha\right) dt \\ &= \tfrac{\sqrt{2}x}{\alpha}(1 - e^{-\alpha}) + \alpha. \end{aligned}$$

If we choose for example $\alpha = \sqrt{\sqrt{2}x}$ the quotient of the lengths of both curves is

$$\frac{\ell(R_{A(t\sqrt{\sqrt{2}x})} \circ \gamma^*)}{\ell(\gamma^*)} = \frac{2 - \exp(-\sqrt{\sqrt{2}x})}{\sqrt{\sqrt{2}x}} \overset{x\to\infty}{\longrightarrow} 0.$$

Hence in some cases horizontal curves can be arbitrarily much longer than some other curves with the same projection and starting point.

Nevertheless in the reductive case it is sufficient to consider the horizontal curves in order to find all boundary points of the Cartan geometry.

Proposition 6.1 *For a Cartan geometry $(\mathcal{G}, \pi, M; \omega)$ modeled on a reductive space $G/_P$ the boundary points are already given by all inextensible, horizontal curves $\gamma : [0,1) \longrightarrow \mathcal{G}$ of finite length.*

Proof: Let $\gamma : [0,1) \longrightarrow \mathcal{G}$ be a curve defining a boundary point of $\mathcal{G}$, that is γ is of finite length and inextendable. We set $\gamma_n : [0,1) \longrightarrow \mathcal{G}$ to be the curve defined to be a part of γ via $\gamma_n(t) := \gamma\left(1 + \frac{t-1}{n}\right)$. For every $n \in \mathbb{N}$ we have a curve $p_n : [0,1) \longrightarrow P$ with $\gamma_n = R_{p_n} \circ \gamma_n^*$ and $p_n(0) = e$, where γ_n^* is the horizontal curve with the same starting point and the same projection as γ_n. We use an inner product on $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{m}$ which corresponds with the splitting, i.e. we require that $\mathfrak{p}$ and $\mathfrak{m}$ are to be orthogonal with respect to the inner product chosen. Now we can write for the length of γ_n:

$$\begin{aligned}
\ell(\gamma_n) &= \int_0^1 \|\omega(\dot{\gamma_n})\| dt \\
&= \int_0^1 \left\|\omega\left(\tfrac{d}{dt}(R_{p_n} \circ \gamma_n^*)\right)\right\| dt \\
&= \int_0^1 \left\|\omega\left(dR_{p_n} \circ \dot{\gamma_n^*} + d\widetilde{L_{p_n^{-1}} \dot{p_n}}\right)\right\| dt \\
&= \int_0^1 \| \underbrace{Ad(p_n^{-1}) \circ \omega(\dot{\gamma_n^*})}_{\in \mathfrak{m}} + \underbrace{dL_{p_n^{-1}} \dot{p_n}}_{\in \mathfrak{p}} \| dt \\
&= \int_0^1 \left(\left\|Ad(p_n^{-1}) \circ \omega(\dot{\gamma_n^*})\right\| + \left\|dL_{p_n^{-1}} \dot{p_n}\right\|\right) dt.
\end{aligned}$$

Since γ is of finite length, the lengths of γ_n tend to zero for increasing n. Hence we have $\dot{p}_n \overset{n\to\infty}{\longrightarrow} 0$ and $p_n(t) \overset{n\to\infty}{\longrightarrow} p_n(0) = e$. So we can find a number $N \in \mathbb{N}$ such that γ_N^* and p_N are of finite length. Since P is a closed group the limit of $p_N(t)$ for $t \to 1$ exists, $\lim_{t\to 1} p_N(t) =: \hat{p} \in P$. Hence γ_N^* has to be inextendable. So $R_{\hat{p}} \circ \gamma_N^*$ is an inextensible, horizontal curve of finite length, defining the same boundary point in $\overline{\mathcal{G}}$ as γ.
Therefore, the boundary points of a Cartan geometry modeled on a reductive space are already given by all inextendable, horizontal curves of finite length.

□

Remark 6.2 *In Section 6.3 we will give an example of a Cartan geometry modeld on a reductive space which has a nonempty Cartan boundary although all ω-constant and consequently all ω-constant horizontal curves are complete. So in the reductive case the set of relevant curves for the Cartan boundary cannot be reduced to the ω-constant horizontal curves also called geodesics.*

6.2.4 The Cartan Boundary of Semi-Riemannian Manifolds

Semi-Riemannian manifolds are a special case of Cartan geometries modeled on a reductive space. The Cartan bundle of a semi-Riemannian manifold (M, g) of signature (p, q) is given by $(\mathcal{O}(M, g), \pi, M; A^{LC} + \theta)$ where $\mathcal{O}(M, g)$ denotes the bundle of all orthonormal reperes with structure group $O(p, q)$.
For semi-Riemannian manifolds geodesics have been used in the past to define boundaries (see for example [Ger68a]) since incomplete geodesics seemed to hint at the presence of singularities. However Geroch himself (see [Ger68]) gave an example of a Lorentzian manifold where geodesical completeness does not imply completeness in an intuitive manner. So geodesical completeness is not a satisfactory concept in the pseudo-Riemannian case. Still any boundary definition is required to produce endpoints for incomplete geodesics.

In order to detect the geodesics of a semi-Riemannian manifold (M,g) in the Cartan bundle $(\mathcal{G}=\mathcal{O}(M,g),\pi,M;A^{LC}+\theta)$ we need the notion of ω-constant curves. A curve $\gamma : I \longrightarrow \mathcal{G}$ is called ω-constant if the Cartan connection transfers its tangent vector to a constant vector in the Lie algebra $\mathfrak{g}$, $\omega(\dot\gamma) = const$.

Lemma 6.3 *Let (M^n,g) be a semi-Riemannian manifold of signature (p,q), $p+q=n$, with the Cartan bundle $(\mathcal{O}(M,g),\pi,M,\omega=\theta+A^{LC})$. The horizontal, ω-constant curves project to geodesics in M and every geodesic in M can be lifted to a horizontal, ω-constant curve. I.e. in $\overline{M}=M\cup\partial_{CB}M$ all geodesics, that is all projections of horizontal, ω-constant curves, are complete.*

Proof: Let $\widetilde{\gamma} : I \longrightarrow \mathcal{O}(M,g)$ be a horizontal ω-constant curve and $\gamma = \pi\circ\widetilde{\gamma}$. Since $\widetilde{\gamma}$ is a repere, we write $\widetilde{\gamma}=(\widetilde{\gamma}_i)_{i=1}^n$. It holds

$$0 = A^{LC}(\dot{\widetilde{\gamma}}) = \sum_{i<j} \varepsilon_i\varepsilon_j g(\nabla_{\dot\gamma}\widetilde{\gamma}_i,\widetilde{\gamma}_j)E_{ij}.$$

Hence we have $\nabla_{\dot\gamma}\widetilde{\gamma}_i = 0$ for all $i=1,\dots,n$. Denote the coordinates of $\dot\gamma$ with respect to the frame $\widetilde{\gamma}$ with γ_k, that is to say $\dot\gamma = \sum_k \varepsilon_k\gamma_k\widetilde{\gamma}_k$. Since $\theta(\dot{\widetilde{\gamma}}) = [\widetilde{\gamma}]^{-1}(\dot\gamma) = const$ we know that all γ_k are constant. Consequently $\frac{\nabla\dot\gamma}{dt} = \sum_k \varepsilon_k\gamma_k\frac{\nabla\widetilde{\gamma}_k}{dt} = 0$ since $\widetilde{\gamma}_k$ is parallel along γ. So γ is a geodesic.
The other way round let $\gamma : I \longrightarrow M$ be a geodesic and $\widetilde{\gamma} : I \longrightarrow \mathcal{O}(M,g)$ a horizontal lift of γ, i.e. $A^{LC}(\dot{\widetilde{\gamma}})=0$. Since $\dot\gamma$ and $\widetilde{\gamma}$ are parallely propagated along γ the displacementform maps $\dot{\widetilde{\gamma}}$ onto a constant vector, $\theta(\dot{\widetilde{\gamma}}) = [\widetilde{\gamma}]^{-1}(\dot\gamma) = const$. Thus $\widetilde{\gamma}$ is a ω-constant horizontal curve.

□

Hence a semi-Riemannian manifold with an empty Cartan boundary is geodesically complete.
Note that not every ω-constant curve in $\mathcal{O}(M,g)$ projects onto a geodesic or pregeodesic. For example let $\widetilde{\gamma}$ be a ω-constant curve with $A^{LC}(\dot{\widetilde{\gamma}}) = E_{kl}$ and $\theta(\dot{\widetilde{\gamma}}) = (1,\dots,1)$. The projection of $\widetilde{\gamma}$ is again denoted with γ. With $A^{LC}(\dot{\widetilde{\gamma}}) = \sum_{i<j}\varepsilon_i\varepsilon_j g(\nabla_{\dot\gamma}\widetilde{\gamma}_i,\widetilde{\gamma}_j)E_{ij} \stackrel{!}{=} E_{kl}$ we obtain $\nabla_{\dot\gamma}\widetilde{\gamma}_k = \varepsilon_k\widetilde{\gamma}_l$ and all other covariant derivatives in direction of $\dot\gamma$ vanish. So we have

$$\begin{aligned}\frac{\nabla\dot\gamma}{dt} &= \frac{\nabla\sum_i\varepsilon_i\widetilde{\gamma}_i}{dt}\\ &= \varepsilon_k\frac{\nabla\widetilde{\gamma}_k}{dt}\\ &= \widetilde{\gamma}_l.\end{aligned}$$

Since $\dot\gamma$ and $\frac{\nabla\dot\gamma}{dt}$ are not collinear γ is no geodesic and no pregeodesic.
Inspired by the two dimensional geodesically complete Lorentzian manifold which contains a timelike curve of bounded acceleration and finite length explained by Geroch in [Ger68] another definition of completeness for Lorentzian manifolds arises (see [BEE96]).

Definition 6.1 *A Lorentzian manifold (M,g) is said to be b.a. complete (bounded acceleration) if all inextensible, unit speed, timelike curves $\gamma : I \longrightarrow M$ with bounded acceleration, that is $|g(\nabla^{LC}_{\dot\gamma}\dot\gamma(t),\nabla^{LC}_{\dot\gamma}\dot\gamma(t))| \le B$ for some $B\in\mathbb{R}$ and all $t\in I$, have infinite length $l=\int_I |g(\dot\gamma,\dot\gamma)|^{\frac{1}{2}}dt$.*

Another example of a Lorentzian manifold which is geodesically complete but not b.a. complete was given in [Beem76].

Lemma 6.4 *A Lorentzian manifold (M,g) which is not b.a. complete has a nontrivial Cartan boundary.*

Proof: Let (M,g) be of signature $(1,q)$. Since the Lorentzian manifold (M,g) is not b.a. complete, we have an inextensible, unit speed, timelike curve $\gamma : I \longrightarrow M$ of bounded acceleration and finite length. This implies especially that the intervall I has to be bounded. We need to prove that the length of γ messured in a parallely propagated orthogonal frame is finit.

Let $\gamma^* : I \longrightarrow \mathcal{O}(M,g)$ be a horizontal lift of γ, that is to say γ^* is a parallely propagated orthogonal repere $(\gamma_0^*, \ldots, \gamma_q^*)$ along γ. We denote the coordinates of the tangent vector $\dot\gamma$ with respect to the frame γ^* with $\dot\gamma_i$, hence $\dot\gamma = \sum_i \dot\gamma_i \gamma_i^*$. With those coordinates we can write

- $g(\dot\gamma,\dot\gamma) = -\dot\gamma_0^2 + \underbrace{\dot\gamma_1^2 + \cdots + \dot\gamma_q^2}_{=:\langle\dot{\hat\gamma},\dot{\hat\gamma}\rangle_q} \overset{!}{=} -1.$

  This implies the existence of a real map $f : I \longrightarrow \mathbb{R}$ with $\langle\dot{\hat\gamma},\dot{\hat\gamma}\rangle_q = \sinh^2 \circ f$ and $\dot\gamma_0^2 = \cosh^2 \circ f$.

- $$\begin{aligned} B &\overset{!}{>} g(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma) \\ &= \textstyle\sum_{ij} g\big(\nabla_{\dot\gamma}(\dot\gamma_i\gamma_i^*), \nabla_{\dot\gamma}(\dot\gamma_j\gamma_j^*)\big) \\ &= \textstyle\sum_{ij} g\big(\dot\gamma_i \underbrace{\nabla_{\dot\gamma}(\gamma_i^*)}_{=0} + \ddot\gamma_i\gamma_i^*, \dot\gamma_j \underbrace{\nabla_{\dot\gamma}(\gamma_j^*)}_{=0} + \ddot\gamma_j\gamma_j^*\big) \\ &= -\ddot\gamma_0^2 + \underbrace{\ddot\gamma_1^2 + \cdots + \ddot\gamma_q^2}_{=\langle\ddot{\hat\gamma},\ddot{\hat\gamma}\rangle_q}. \end{aligned}$$

According to the Cauchy Schwarz inequality we get

$$\begin{aligned} \langle\ddot{\hat\gamma},\ddot{\hat\gamma}\rangle_q &\geq \frac{\langle\ddot{\hat\gamma},\dot{\hat\gamma}\rangle_q^2}{\langle\dot{\hat\gamma},\dot{\hat\gamma}\rangle_q} \\ &= \left(\frac{d}{dt}\sqrt{\langle\dot{\hat\gamma},\dot{\hat\gamma}\rangle_q}\right)^2 \\ &= \left(\frac{d}{dt}\sqrt{\sinh^2 \circ f}\right)^2 \\ &= \dot f^2 \cosh^2 \circ f. \end{aligned}$$

Since γ is a curve of bounded acceleration, the derivative of f has to be bounded as well since we can write using the estimation above

$$\begin{aligned} B &> g(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma) \\ &\geq -\dot f^2 \sinh^2 \circ f + \dot f^2 \cosh^2 \circ f \\ &= \dot f^2. \end{aligned}$$

Hence f is bounded on the bounded intervall I and so is the displacement form applied to the derivative of the horizontal lift γ^*.

$$\begin{aligned} \|\theta(\dot\gamma^*)\|^2 &= \left\|[\gamma^*]^{-1}(\dot\gamma)\right\|^2 \\ &= \dot\gamma_0^2 + \dot\gamma_1^2 + \cdots + \dot\gamma_q^2 \\ &= \cosh^2 \circ f + \sinh^2 \circ f \end{aligned}$$

However this implies that the length of γ^* is finite, since the intervall I and the map f are bounded. Of cause γ^* is inextendable for otherwise its projection γ could be extended. Hence γ^* defines a boundary point in the Cartan bundle $\mathcal{O}(M,g)$. So any Lorentzian manifold which is not b.a. complete has a nontrivial Cartan boundary.

□

Remark 6.3 *Spacelike geodesic completeness is not implied by b.a. completeness. An example is given by changing the sign of the metric of the manifold given in [Ger68] which is timelike incomplete but null and spacelike complete.*

Remark 6.4 *The concept of b.a. completeness does not work for pseudo-Riemannian manifolds with more than one time dimension. Here a unit length, timelike curve of finite length and bounded acceleration may have infinite length messured in a parallely propagated frame, i.e. a horizontal lift of such a curve may be of infinite length. Thus a pseudo-Riemannian manifold with more than one time dimension might have an empty Cartan boundary and still be b.a. incomplete.*
Consider for example in the flat pseudo-Riemannian space $\mathbb{R}^{2,1}$ the following inextensible curve $\gamma : (0,1] \longrightarrow \mathbb{R}^{2,1}$, $\gamma(t) = \left(t, e^{\frac{1}{t}}, e^{\frac{1}{t}}\right)$. Due to the signature the second and third coordinates cancel out. So the length of the curve γ is one, $l = \int_0^1 |g(\dot\gamma,\dot\gamma)|^{\frac{1}{2}} dt = \int_0^1 1dt = 1$. And its acceleration is bounded since

$$\begin{aligned} & |g(\nabla^{LC}_{\dot\gamma}\dot\gamma(t), \nabla^{LC}_{\dot\gamma}\dot\gamma(t))| \\ = \; & |\langle\gamma''(t), \gamma''(t)\rangle_{2,1}| \\ \equiv \; & 0. \end{aligned}$$

At the same time the length of the horizontal lift $\gamma^*(t) = (e_1, e_2, e_3)_{\gamma(t)}$ is infinit

$$\begin{aligned} \ell(\gamma^*) &= \int_0^1 \|\theta(\dot{\gamma^*}(t))\| dt \\ &= \int_0^1 \left\|[(e_1,e_2,e_3)]^{-1}\big(\dot\gamma(t)\big)\right\| dt \\ &= \int_0^1 \sqrt{\tfrac{2}{t^4}e^{\frac{2}{t}} + 1} dt \\ &> \int_0^1 \sqrt{\tfrac{2}{t^4}e^{\frac{2}{t}}} dt \\ &= \int_0^1 \tfrac{\sqrt{2}}{t^2} e^{\frac{1}{t}} dt \\ &= \left[-\sqrt{2}e^{\frac{1}{t}}\right]_0^1 \\ &= \infty. \end{aligned}$$

In addition it seems weird to call such a curve one of bounded acceleration. So this concept should be restriced to the Lorentzian case.

6.2.5 The Cartan Boundary of Riemannian Manifolds

As we will now see, for Riemannian manifolds the Cartan boundary is exactly the same as the boundary defined by the metric. I.e. the Cartan boundary is a generalisation of the metrical boundary.
The Cartan bundle of a Riemannian manifold (M,g) is given by $(\mathcal{O}(M,g), \pi, M; A^{LC} + \theta)$ with structure group the connected component of $O(n)$ containing the neutral element, which we will denote by $O_c(n)$.

Proposition 6.2 *For a Riemannian manifold (M,g) the Cartan boundary is identical to the metrical boundary defined by the metric g.*

Proof: As we have seen the points of the Cartan boundary are given by all inextensible, horizontal cuves of finite length. Let $\gamma : [0,1) \longrightarrow \mathcal{O}(M,g)$ be an inextendable, horizontal curve of finite length. We can write

$$\begin{aligned}
\ell(\gamma) &= \int_0^1 \|\omega(\dot\gamma)\| dt \\
&= \int_0^1 \|\theta(\dot\gamma) + \underbrace{A^g(\dot\gamma)}_{=0}\| dt \\
&= \int_0^1 \|[\gamma]^{-1}(d\pi \circ \dot\gamma)\| dt \\
&= \int_0^1 \sqrt{g\big(d\pi(\dot\gamma), d\pi(\dot\gamma)\big)} dt \\
&= \ell_g(\pi \circ \gamma).
\end{aligned}$$

So every inextendable, horizontal curve of finite length in $\mathcal{O}(M,g)$ projects to an inextensible curve of finite length in M. The other way round, the horizontal lift of an inextensible curve in M is also of finite length. Hence the metrical boundary defined by the Riemannian metric g is identical to the Cartan boundary.

□

Remark 6.5 *In [NO61] it was proven that any Riemannian manifold can be made geodesically complete by a global conformal change. So, since in the Riemannian case geodesical completeness and Cartan completeness are equivalent, this is also true for the Cartan boundary. I.e. for any Riemannian manifold we have a global conformal change such that it has no Cartan boundary.*

Remark 6.6 *In the pseudo-Riemannian case there are manifolds which are nonspacelike geodesically incomplete and cannot be made complete by any global conformal factor. See for example [Mis67].*

6.3 Completeness

As we have seen in the examples in Section 6.2 there are several ways of defining completeness. We say that a manifold is Cartan complete if its Cartan boundary is empty. A manifold with a Cartan geometry modeled on a reductive space is said to be horizontally complete if all horizontal curves in the total space of the Cartan bundle are complete. For Lorentzian manifolds we also have the notion of b.a. completeness, that is all unit speed, timelike curves of bounded acceleration have to be complete. And for semi-Riemannian manifolds (spacelike, timelike, null) geodesical completness is given if all (spacelike, timelike, null) geodesics are complete. We will at first illustrate the relations between those concepts as already discussed above.
For a manifold with a Cartan geometry modeled on a reductive space Cartan completeness and horizontal completeness are equivalent and imply ω-completeness.

$$\begin{array}{ccc}
\text{Cartan completeness} & \Longleftrightarrow & \text{horizontal completeness} \\
\Downarrow \not\Uparrow & & \\
\omega - \text{completeness} & & \\
\text{(defined later)} & &
\end{array}$$

For Lorentzian manifolds we have many different concepts of completeness which are not equivalent.

$$\begin{array}{ccc}
\text{Cartan completeness} & \Longleftrightarrow & \text{horizontal completeness} \\
\Downarrow\not\Uparrow & & \Downarrow\not\Uparrow \\
\text{b.a. completeness} & \not\Longleftrightarrow & \text{geodesical completeness} \\
\Downarrow\not\Uparrow & & \\
\begin{array}{c}\text{timelike geodesical}\\ \text{completeness}\end{array} & &
\end{array}$$

In the Riemannian case all concepts coincide.

$$\begin{array}{ccc}
\text{Cartan completeness} & \Longleftrightarrow & \text{horizontal completeness} \\
\Updownarrow & & \Updownarrow \\
\omega\text{-completeness} & \Longleftrightarrow & \text{geodesical completeness}
\end{array}$$

We will now discuss concepts of completeness in the general setting.
Let $(\mathcal{G}, \pi, M; P)$ be a Cartan bundle with structure group P, $LA(P) = \mathfrak{p}$, and the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$.
Although we can define horizontal subspaces $\mathcal{H}_u \subset T_u\mathcal{G}$ in the Cartan bundle with the help of the Cartan connection via

$$pr_{\mathfrak{p}} \circ \omega(\mathcal{H}_u) \stackrel{!}{=} 0,$$

where we project with respect to the fixed frame $(a_1, \ldots, a_n)$ of $\mathfrak{g}$ which is chosen such that $(a_1, \ldots, a_r)$ is a frame of $\mathfrak{p}$, those subspaces do not merge to a subbundle nor do they define a bundle connection, since they are not P-invariant.
We have $pr_{\mathfrak{p}} \circ \omega \circ dR_p(\mathcal{H}_u) = pr_{\mathfrak{p}} \circ Ad(p^{-1}) \circ \omega(\mathcal{H}_u)$ and the projection and the adjoint action usually do not commute. So in the general case horizontal lifts of curves in the base manifold M do not necessarily exist for the whole curves. Hence we have to drop the concept of horizontal completeness for Cartan geometries not modeled on reductive spaces.
Literature sugests that one might still consider only horizontal curves, i.e. curves $\gamma : I \longrightarrow \mathcal{G}$ with $pr_{\mathfrak{p}} \circ \omega(\dot\gamma) = 0$ with respect to a chosen frame, in order to determine a Cartan boundary, arguing that for an inextendable curve $\gamma : [0, 1) \longrightarrow \mathcal{G}$ of finite length the development of this curve, that is to say the curve $p_\gamma : [0, 1) \longrightarrow G$ with $\omega_G(\dot{p_\gamma}) = \omega(\dot\gamma)$, is a curve of finite length in a complete space and thus can be extended. Then it is claimed that there would be a small intervall $[1 - \varepsilon, 1]$ and a map $p : [1 - \varepsilon, 1] \longrightarrow P$ such that $R_p p_\gamma|_{[1-\varepsilon,1]}$ and consequently also the curve $R_p\gamma|_{[1-\varepsilon,1]}$ would be horizontal. However we would like to point out that the extended curve p_γ is continuous but not necessarily differentiable. Thus the existence of the curve $p : [1 - \varepsilon, 1) \longrightarrow P$ making γ horizontal is not guaranteed.
However there is a natural set of special curves in the Cartan bundle, the ω-constant curves, yielding the definition of ω-completeness already mentioned. For $X \in \mathfrak{g}$ we define the ω-constant vector field related to X by

$$\begin{array}{rcl}
\omega^{-1}(X) : \mathcal{G} & \longrightarrow & T\mathcal{G} \\
u & \mapsto & \omega_u^{-1}(X).
\end{array}$$

For $X \in \mathfrak{p}$ the ω-constant vector fields are the fundamental vector fields $\omega^{-1}(X) = \widetilde{X}$ and their integral curves through a point $u \in \mathcal{G}$ are $\phi_u^{\widetilde{X}}(t) = R_{\exp(tX)}u$. Hence the ω-constant vector fields related to elements of $\mathfrak{p}$ are complete, since their integral curves are defined on $\mathbb{R}$ and have infinite length.

Definition 6.2 *A Cartan geometry $(\mathcal{G}, \pi, M; \omega)$ is called ω-complete if all ω-constant vector fields are complete.*

Since the vertical ω-constant curves, i.e. the integral curves of the fundamental vector fields, are always complete the question arises, wether ω-completeness is already given if all horizontal ω-constant vector fields are complete.

Definition 6.3 *A Cartan geometry $(\mathcal{G}, \pi, M; \omega)$ is called geodesically complete if all horizontal, ω-constant vector fields are complete.*

Although we have for ω-constant curves several familiar properties know from the geodesics of Riemannian manifolds such as

- For every point u of $\mathcal{G}$ there is an ω-normal neighbourhood such that every point within the neighbourhood can be connected with u by a unique ω-constant curve within this ω-normal neighbourhood.

- For $\mathcal{G}$ being connected and any points $u, v \in \mathcal{G}$ there is a piecewise ω-constant curve γ connecting u and v.

- If the limit $\lim_{t\to b}$ of an ω-constant curve defined on some bounded intervall $(a, b) \subset \mathbb{R}$ exists in $\mathcal{G}$ there is a $\varepsilon > 0$ such that the curve can be prolonged to an ω-constant curve on $(a, b + \varepsilon)$.

As in the pseudo-Riemannian case essential properties ensuring the theorem of Hopf and Rinow are missing. ω-constant curves are not locally minimizing. Furthermore if all ω-constant curves starting at a point $u \in \mathcal{G}$ are complete, the question is whether there might still be points in $\mathcal{G}$ which cannot be connected to u by an ω-constant curve.
Obviously we have the following hierarchy between the concepts of completeness given and actually non of those are equivalent as we are about to see.

$$\begin{array}{c} \text{Cartan completeness} \\ \Downarrow \\ \omega\text{-completeness} \\ \Downarrow \\ \text{geodesical completeness} \end{array}$$

6.3.1 First Example

We now want to give an example of a Cartan geometry which is ω-complete but not Cartan complete. This example was constructed by Y.Clifton in [Cli66].
Consider the following Cartan geometry $(\mathcal{G} = \mathbb{R}^2 \setminus \{0\}, id, M = \mathbb{R}^2 \setminus \{0\})$, that is a trivial $\{id\}$-principal bundle, of type $(G = \mathbb{R}^2, P = \{id\})$. Please note, that this is actually a Cartan geometry modeled on a (trivial) reductive space. All curves in $\mathcal{G}$ are already horizontal.
In polar coordinates the Cartan connection ω is given by

$$\begin{array}{rcl} \omega : T\mathcal{G} & \longrightarrow & \mathfrak{g} = \mathbb{R}^2 \\ \left(\frac{\partial}{\partial r}\right)_{(r,\varphi)} & \mapsto & (\cos\frac{1}{r}, -\sin\frac{1}{r}) \\ \left(\frac{\partial}{\partial \varphi}\right)_{(r,\varphi)} & \mapsto & (r\sin\frac{1}{r}, r\cos\frac{1}{r}). \end{array}$$

For every point (r, φ) this is an isomorphism $\omega : T_{(r,\varphi)}\mathcal{G} \longrightarrow \mathfrak{g}$. Since the group P is trivial ω is also right invariant under the group action and the generators of the fundamental vector fields are trivially reproduced. So ω is a Cartan connection. A global basis is given by

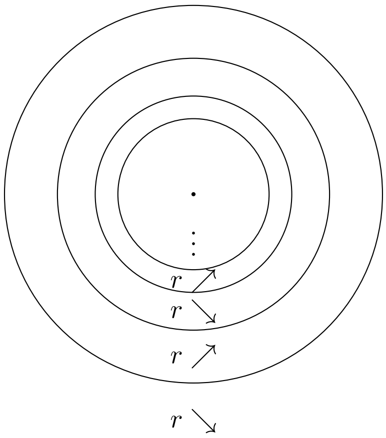
$e_1 := \omega^{-1}(1,0) = (\cos\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r}\sin\frac{1}{r}\frac{\partial}{\partial\varphi})$ and $e_2 := \omega^{-1}(0,1) = (-\sin\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r}\cos\frac{1}{r}\frac{\partial}{\partial\varphi})$. So the ω-constant vector field of $(\alpha,\beta) \in \mathfrak{g}$ is

$$X^{(\alpha,\beta)}(r,\varphi) = \alpha e_1(r,\varphi) + \beta e_2(r,\varphi).$$

Hence for an ω-constant curve $\gamma = (r,\varphi)$, defined by $\dot\gamma = \dot r\frac{\partial}{\partial r} + \dot\varphi\frac{\partial}{\partial\varphi} \overset{!}{=} \alpha e_1 + \beta e_2$, the angle φ is completely determined by the radius r and the initial angle $\varphi(0)$, $\dot\varphi \overset{!}{=} \frac{\alpha}{r}\sin\frac{1}{r} + \frac{\beta}{r}\cos\frac{1}{r}$. The radius coordinate has to fullfill $\dot r \overset{!}{=} \alpha\cos\frac{1}{r} - \beta\sin\frac{1}{r}$. This means that for fixed (α,β) the punctured plane $\mathbb{R}^2 \setminus \{0\}$ is devided by the circles with radius

$$\begin{array}{rll} & r = \frac{1}{\arctan\frac{\alpha}{\beta} + k\pi}, & k \in \mathbb{N} \text{ such that } r > 0 \quad \text{if } \beta \neq 0 \\ \text{or} & r = \frac{1}{\frac{\pi}{2} + k\pi}, & k \in \mathbb{N} \text{ such that } r > 0 \quad \text{if } \beta = 0 \end{array}$$

into regions with $\dot r > 0$ or $\dot r < 0$ and $\dot r = 0$ exactly on the circles.



Hence an ω-constant curve starting in one of those regions will never leave this region. More precisely ω-constant curves starting within the biggest circle stay within this circle and since they cannot pass one of the other circles they cannot get arbitrarily close to zero. They are trapped within a compact set and are therefore complete. All ω-constant curves starting on one of the circles will stay on this circle and are terefore complete. Now outside the biggest cirlce the derivative of the radius coordinate is bounded, $\dot r \in (-|\alpha| - |\beta|, |\alpha| + |\beta|)$. So an ω-constant curve starting outside the biggest circle cannot enter this circle and its radius coordinate cannot go to infinity within a finite amount of time. Consequently these curves as well are complete.

Since the arguments above are true for all $(\alpha,\beta) \in \mathfrak{g}$ we conclude that all ω-constant vector fields are complete, that is to say the given Cartan geometry is ω-complete.

However the curve $\gamma : (0,1] \longrightarrow M = \mathbb{R}^2 \setminus \{0\}$ being defined by $\gamma(t) = (t,0)$ cannot be prolonged for $t \to 0$ and has finite length $\ell(\gamma) = \int_0^1 \|\omega(\dot\gamma)\|dt = \int_0^1 \|(\cos\frac{1}{t}, -\sin\frac{1}{t})\|dt = 1$. So the Cartan boundary of M is not empty and therefore M is not Cartan complete although it is ω-complete.

$$\begin{array}{c}\text{Cartan completeness} \\ \Downarrow \not\Uparrow \\ \omega\text{-completeness}\end{array}$$

6.3.2 Second Example

We will now give an example of a Cartan geometry which is geodesically complete but not ω-complete. The idea of this example is taken from [Cli66]. Our example will be of type $(L_2, Gl(2))$, where L_2 is the group of all linear transformations of the two dimensional plane $\mathbb{R}^2 = \left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \middle| \; x \in \mathbb{R}^2 \right\}$,

$$L_2 \;:=\; \left\{ \begin{pmatrix} 1 & 0 \\ w & A \end{pmatrix} \middle| \; A \in Gl(2), w \in \mathbb{R}^2 \right\} \subset Gl(3).$$

Its Lie algebra is

$$\mathfrak{l}_2 = \left\{ \begin{pmatrix} 0 & 0 \\ w & A \end{pmatrix} \middle| \; A \in M(2\times 2), w \in \mathbb{R}^2 \right\} = \mathfrak{gl}(2) \oplus \mathbb{R}^2.$$

In order to simplify the notation we will write

$$\mathfrak{l}_2 = \left\{ w \oplus A \mid \; A \in M(2\times 2), w \in \mathbb{R}^2 \right\} = \mathfrak{gl}(2) \oplus \mathbb{R}^2.$$

The group $Gl(2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \middle| \; g \in Gl(2) \right\}$ of all general linear transformations acts on $\mathfrak{l}_2$ by the Adjoint action

$$\begin{aligned} Ad\left(\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ w & A \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ gw & gAg^{-1} \end{pmatrix} \\ \text{or rather} \qquad Ad(g)(w \oplus A) &= gw \oplus Ad(g)A \qquad \text{for all } g \in Gl(2), w \oplus A \in \mathfrak{l}_2. \end{aligned}$$

Note that $Gl(2)$ acts by isomorphisms on $\mathfrak{l}_2$. Obviously $L_2/_{Gl(2)}$ is a reductive space since both subalgebras of $\mathfrak{l}_2 = \mathfrak{gl}(2) \oplus \mathbb{R}^2$ are invariant under the Adjoint action of $Gl(2)$.
Consider the manifold $M := \mathbb{R}^2$ and the trivial $Gl(2)$ principal bundle $\mathcal{G} := M \times Gl(2)$. We have a global section

$$\begin{aligned} \sigma : M &\longrightarrow \mathcal{G} \\ x &\mapsto (x, id) = \begin{pmatrix} 1 & 0 \\ x & I_2 \end{pmatrix}. \end{aligned}$$

The Cartan connection on $(\mathcal{G}, \pi, M)$ is defined with the help of the section σ:

$$\begin{aligned} \sigma^*\omega(e_1) &:= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ \sigma^*\omega(e_2) &:= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

ω is defined to meet the claims of equivariance and reproduction of the generators of the fundamental vector fields. This is an isomorphism for every point $u \in \mathcal{G}$, since according to the definition we have for the point $\sigma(x)$

$$\begin{aligned} T_{\sigma(x)}\mathcal{G} &= Tv_{\sigma(x)}\mathcal{G} \oplus \mathbb{R}d\sigma(e_1) \oplus \mathbb{R}d\sigma(e_2) \\ &\overset{\omega}{\mapsto} \mathfrak{gl}(2) \oplus \mathbb{R}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) \oplus \mathbb{R}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \\ &= \mathfrak{l}_2. \end{aligned}$$

And with $Gl(2)$ acting by isomorphisms on $\mathfrak{l}_2$ the Cartan connection defined above is actually an isomorphism for all points $u \in \mathcal{G}$ as requested.

The geodesics of the Cartan geometry $(\mathcal{G},\pi,M;\omega)$ are the curves $\gamma : I \longrightarrow M$ which admit a horizontal, ω-constant lift $\hat{\gamma} : I \longrightarrow \mathcal{G}$. So in order to show that $(\mathcal{G},\pi,M,\omega)$ is geodesically complete we have to find all curves $\gamma : I \longrightarrow M$ and $g : I \longrightarrow Gl(2)$ such that $\omega\left(\frac{d}{dt}R_g\circ\sigma\circ\gamma\right) = \begin{pmatrix} a \\ b \end{pmatrix} \oplus 0$ for any constants $a,b\in\mathbb{R}$.

We start with

$$\begin{aligned}
\omega\left(\tfrac{d}{dt}R_g\circ\sigma\circ\gamma\right) &= \omega\left(dR_g\circ d\sigma\circ\dot{\gamma} + \widetilde{dL_{g^{-1}}\dot{g}}\right)\\
&= Ad(g^{-1})\circ(\sigma^*\omega)(\dot{\gamma}) + dL_{g^{-1}}\dot{g}\\
&= Ad(g^{-1})\left(\begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \end{pmatrix} \oplus \begin{pmatrix} 0 & -\dot{\gamma}_2 \\ \dot{\gamma}_2 & \dot{\gamma}_1 \end{pmatrix}\right) + dL_{g^{-1}}\dot{g}\\
&\overset{!}{=} \begin{pmatrix} a \\ b \end{pmatrix}.
\end{aligned}$$

So with $g = \begin{pmatrix} \alpha & \beta \\ \eta & \delta \end{pmatrix}$ we find

$$\begin{aligned}
\dot{\gamma}_1 &= a\alpha + b\beta \ , \ \dot{\alpha} = \dot{\gamma}_2\eta \ , \ \dot{\eta} = -\dot{\gamma}_2\alpha - \dot{\gamma}_1\eta \text{ and}\\
\dot{\gamma}_2 &= a\eta + b\delta \ , \ \dot{\beta} = \dot{\gamma}_2\delta \ , \ \dot{\delta} = -\dot{\gamma}_2\beta - \dot{\gamma}_1\delta.
\end{aligned} \qquad (*)$$

Especially we have

$$\begin{aligned}
a\dot{\alpha} + b\dot{\beta} &= \dot{\gamma}_2\underbrace{(a\eta + b\delta)}_{=\dot{\gamma}_2} &&, \text{ i.e. } \ddot{\gamma}_1 = \dot{\gamma}_2^2,\\
\text{and}\quad a\dot{\eta} + b\dot{\delta} &= -\dot{\gamma}_2\underbrace{(a\alpha + b\beta)}_{=\dot{\gamma}_1} - \dot{\gamma}_1\underbrace{(a\eta + b\delta)}_{=\dot{\gamma}_2} &&, \text{ i.e. } \ddot{\gamma}_2 = -2\dot{\gamma}_1\dot{\gamma}_2.
\end{aligned}$$

Hence all curves γ fulfilling the equations above also satisfy

$$\begin{aligned}
0 &= 2\dot{\gamma}_1\ddot{\gamma}_1 + \dot{\gamma}_2\ddot{\gamma}_2\\
&= \tfrac{d}{dt}\left(\dot{\gamma}_1^2 + \tfrac{1}{2}\dot{\gamma}_2^2\right)\\
&= \tfrac{d}{dt}\left(\dot{\gamma}_1^2 + \tfrac{1}{2}\ddot{\gamma}_1\right).
\end{aligned}$$

So we can conclude $\ddot{\gamma}_1 = const - 2\dot{\gamma}_1^2$. And with $\ddot{\gamma}_1 = \dot{\gamma}_2^2 \geq 0$ we obtain either $\dot{\gamma}_1 = \dot{\gamma}_2 = 0$, which would be a stationary curve, or $\ddot{\gamma}_1 = c^2 - 2\dot{\gamma}_1^2$ with $c > 0$. In this case we obtain

$$\begin{aligned}
\dot{\gamma}_1 &= \tfrac{c}{\sqrt{2}}\tanh\left(\sqrt{2}ct + d\right) &&, \quad \dot{\gamma}_2 = \tfrac{\pm c}{\cosh(\sqrt{2}ct + d)}\\
\text{or}\quad \dot{\gamma}_1 &= \tfrac{c^2}{2} &&, \quad \dot{\gamma}_2 = 0.
\end{aligned}$$

Now we have to find a curve $g : \mathbb{R} \longrightarrow Gl(2)$ making $R_g\circ\sigma\circ\gamma$ horizontal and ω-constant.

Case 1 $b = 0$

Let us at first take a look at the case $b = 0$. Then a has to be nonzero. According to $(*)$ we have $\dot{\gamma}_1 = a\alpha$ and $\dot{\gamma}_2 = a\eta$. Furthermore β and δ have to satisfy:

$$\dot{\beta} = \delta\dot{\gamma}_2 \text{ and } \dot{\delta} = -\beta\dot{\gamma}_2 - \delta\dot{\gamma}_1.$$

Case 1.1 $\dot{\gamma}_1 = \frac{c^2}{2}$ and $\dot{\gamma}_2 = 0$

In this case the differential equations for β and δ simplify a lot. We obtain

$$\dot{\beta} = 0 \text{ and } \dot{\delta} = -\frac{c^2}{2}\delta.$$

I.e. $\beta = const$ and $\delta = d\cdot\exp(-\frac{c^2}{2}t)$. Thus the matrix g is

$$g = \begin{pmatrix} \frac{c^2}{2a} & \beta \\ 0 & d\cdot\exp(-\frac{c^2}{2}t) \end{pmatrix} \in Gl(2) \text{ for } \beta,c,d\in\mathbb{R}, c,d\neq 0.$$

Case 1.2 $\dot\gamma_1 = \frac{c}{\sqrt{2}}\tanh(\sqrt{2}ct+d)$ and $\dot\gamma_2 = \pm\frac{c}{\cosh(\sqrt{2}ct+d)}$

Keeping in mind that we require $g \in Gl(2)$, the map β may not be a multiple of $\dot\gamma_1$ and δ may not be a multiple of $\dot\gamma_2$. This leads to the following equation for β:

$$\dddot\beta + 3\dot\gamma_1\dot\beta + \dot\gamma_2^2\beta = 0.$$

One special solution of this equation is $\beta_1 = \dot\gamma_1$, which however is no solution for the whole problem as mentioned above. Making the Ansatz $\beta = const \cdot e^{\alpha(t)}$ leads to the Riccatti differential equation $\ddot\alpha + \dot\alpha^2 + 3\dot\gamma_1\dot\alpha + \dot\gamma_2^2 = 0$ with one special solution $\alpha_s = \log(\dot\gamma_1)$. Although this Ansatz can only be made for $\beta \neq 0$ and $t > -\frac{d}{\sqrt{2}c}$ it will lead to a solution which can be prolonged.

In order to solve the Ricatti differential equation we set $\dot\alpha = \dot\alpha_s + u$ and have to solve $\dot u = (-2\dot\alpha_s - 3\dot\gamma_1)u - u^2$. We substitute $x = \frac{1}{u}$ to obtain the differential equation $\dot x = (2\dot\alpha_s + 3\dot\gamma_1)x + 1$ with the solution

$$x = \left(\int e^{-2\alpha_s - 3\gamma_1}dt + const\right)\cdot e^{2\alpha_s+3\gamma_1}.$$

Hence we have

$$\begin{aligned}
\dot\alpha &= \dot\alpha_s + \tfrac{1}{x}\\
&= \tfrac{d}{dt}\log(\dot\gamma_1) + \left(\int e^{-2\log(\dot\gamma_1)-3\gamma_1}dt + const\right)^{-1}\cdot e^{-2\log(\dot\gamma_1)-3\gamma_1}\\
&= \tfrac{d}{dt}\log(\dot\gamma_1) + \tfrac{d}{dt}\log\left(\int e^{-2\log(\dot\gamma_1)-3\gamma_1}dt + const\right)\\
&= \tfrac{d}{dt}\log(\dot\gamma_1) + \tfrac{d}{dt}\log\left(\int \dot\gamma_1^{-2}e^{-3\gamma_1}dt + const\right).
\end{aligned}$$

So we obtain another solution for β for $t > -\frac{d}{\sqrt{2}c}$

$$\begin{aligned}
\beta_2(t) &= \tfrac{1}{c^2}\cdot e^{\alpha}\\
&= \tfrac{1}{c^2}\dot\gamma_1\int_\infty^t \dot\gamma_1^{-2}e^{-3\gamma_1}du\\
&= \tfrac{1}{\sqrt{2}c}\tanh(\sqrt{2}ct+d)\int_\infty^t \frac{\sqrt{\cosh(\sqrt{2}cu+d)}}{\sinh^2(\sqrt{2}cu+d)}du\\
&= \tanh(\sqrt{2}ct+d)\int_\infty^{\sqrt{2}ct+d}\frac{\sqrt{\cosh(u)}}{\sinh^2(u)}du.
\end{aligned}$$

Since it holds $\frac{1}{\sinh^2(u} \le \frac{\sqrt{\cosh(u)}}{\sinh^2(u)} \le \frac{\cosh(u)}{\sinh^2(u)}$ we have

$$\begin{array}{rlll}
& \tanh(x) - 1 & & \\
= & \tanh(x)(-\coth(x)+1) & & \\
= & \tanh(x)\int_\infty^x \frac{1}{\sinh^2(u)}du & \ge \tanh(x)\int_\infty^x \frac{\sqrt{\cosh(u)}}{\sinh^2(u)}du & \ge \tanh(x)\int_\infty^x \frac{\cosh(u)}{\sinh^2(u)}du\\
& & & = \tanh(x)(-\frac{1}{\sinh(x)})\\
& & & = -\frac{1}{\cosh(x)}\\
& \downarrow x \to 0 & & \downarrow x \to 0\\
& -1 & & -1
\end{array}.$$

So for $t \searrow -\frac{d}{\sqrt{2}c}$ the map β_2 tends to -1.

We denote with $\widetilde{c}$ the limit of the derivative of β_2,

$$\widetilde{c} := \lim_{t\searrow -\frac{d}{\sqrt{2}c}} \dot\beta_2(t).$$

This limit actually exists as we will check now. With $x = \sqrt{2}ct + d$ we can write for the first derivative of β_2:

$$\begin{aligned}
\dot{\beta}_2(x) &= \tfrac{d}{dt}\left(\tanh x \int_\infty^x \tfrac{\sqrt{\cosh u}}{\sinh^2 u} du\right) \\
&= \tfrac{\sqrt{2}c}{\cosh^2 x}\int_\infty^x \tfrac{\sqrt{\cosh u}}{\sinh^2 u} du + \sqrt{2}c \tanh x \tfrac{\sqrt{\cosh x}}{\sinh^2 x} \\
&= \tfrac{\sqrt{2}c}{\cosh^2 x}\int_\infty^x \tfrac{\sqrt{\cosh u}}{\sinh^2 u} du + \tfrac{\sqrt{2}c}{\sinh x \sqrt{\cosh x}} \\
&= \tfrac{\sqrt{2}c}{\sinh x \cosh x}\left(\beta_2(x) + \sqrt{\cosh x}\right).
\end{aligned}$$

This is bounded since

$$\begin{aligned}
1 &\overset{x\to 0}{\longleftarrow} \tfrac{\sinh x - \cosh x + \sqrt{\cosh x}^3}{\cosh^2 x \sinh x} \\
&= \tfrac{1}{\cosh^2 x}(1 - \coth x) + \tfrac{1}{\sinh x \sqrt{\cosh x}} \\
&= \tfrac{1}{\cosh^2 x}\int_\infty^x \tfrac{1}{\sinh^2 u} du + \tfrac{1}{\sinh x \sqrt{\cosh x}} \\
&\geq \tfrac{1}{\cosh^2 x}\int_\infty^x \tfrac{\sqrt{\cosh u}}{\sinh^2 u} du + \tfrac{1}{\sinh x \sqrt{\cosh x}} \\
&\geq \tfrac{1}{\cosh^2 x}\int_\infty^x \tfrac{\cosh u}{\sinh^2 u} du + \tfrac{1}{\sinh x \sqrt{\cosh x}} \\
&= \tfrac{1}{\cosh^2 x}\left(-\tfrac{1}{\sinh x}\right) + \tfrac{1}{\sinh x \sqrt{\cosh x}} \\
&= \tfrac{\sqrt{\cosh x}^3 - 1}{\cosh^2 x \sinh x} \\
&\overset{x\to 0}{\longrightarrow} 0.
\end{aligned}$$

If we know that the limit of the second derivative of β_2 actually exists and is final, the existence of $\widetilde{c} := \lim_{t \searrow -\frac{d}{\sqrt{2}c}} \dot{\beta}_2(t)$ is clear. So we have to take a look at the second derivative of β_2.

$$\begin{aligned}
\ddot{\beta}_2(x) &= \tfrac{d}{dt}\left(\tfrac{\sqrt{2}c}{\sinh x \cosh x}\left(\beta_2(x) + \sqrt{\cosh x}\right)\right) \\
&= -\tfrac{2c^2(\sinh^2 x + \cosh^2 x)}{\sinh^2 x \cosh^2 x}\left(\beta_2(x) + \sqrt{\cosh x}\right) + \tfrac{\sqrt{2}c}{\sinh x \cosh x}\left(\dot{\beta}_2(x) + \tfrac{\sqrt{2}c \sinh x}{2\sqrt{\cosh x}}\right) \\
&= -\tfrac{2c^2(\sinh^2 x + \cosh^2 x)}{\sinh^2 x \cosh^2 x}\left(\beta_2(x) + \sqrt{\cosh x}\right) \\
&\quad + \tfrac{\sqrt{2}c}{\sinh x \cosh x}\left(\tfrac{\sqrt{2}c}{\sinh x \cosh x}\left(\beta_2(x) + \sqrt{\cosh x}\right) + \tfrac{\sqrt{2}c \sinh x}{2\sqrt{\cosh x}}\right) \\
&= 2c^2 \tfrac{1 - (\sinh^2 x + \cosh^2 x)}{\sinh^2 x \cosh^2 x}\left(\beta_2(x) + \sqrt{\cosh x}\right) + \tfrac{c^2}{\sqrt{\cosh x}^3} \\
&= -4c^2 \tfrac{\sinh^2 x}{\sinh^2 x \cosh^2 x}\left(\beta_2(x) + \sqrt{\cosh x}\right) + \tfrac{c^2}{\sqrt{\cosh x}^3} \\
&= -\tfrac{4c^2}{\cosh^2 x}\left(\beta_2(x) + \sqrt{\cosh x}\right) + \tfrac{c^2}{\sqrt{\cosh x}^3} \\
&\overset{x\to 0}{\longrightarrow} -4c^2(-1+1) + c^2 = c^2
\end{aligned}$$

Hence $\widetilde{c} := \lim_{t \searrow -\frac{d}{\sqrt{2}c}} \dot{\beta}_2(t)$ exists and is finite and we can set

$$\beta_2(t) = \begin{cases} \tanh(\sqrt{2}ct+d)\int_{\infty}^{\sqrt{2}ct+d} \frac{\sqrt{\cosh(u)}}{\sinh^2(u)}du & \text{for } t > \frac{-d}{\sqrt{2}c}, \\ -1 & \text{for } t = \frac{-d}{\sqrt{2}c}, \\ \frac{2\widetilde{c}}{\sqrt{2}c}\tanh(\sqrt{2}ct+d) + \tanh(\sqrt{2}ct+d)\int_{-\infty}^{\sqrt{2}ct+d} \frac{\sqrt{\cosh(u)}}{\sinh^2(u)}du & \text{for } t < \frac{-d}{\sqrt{2}c}. \end{cases}$$

This is a C^1-map and all solutions for the differential equation $\ddot{\beta}+3\dot{\gamma}_1\dot{\beta}+\dot{\gamma}_2^2\beta = 0$ are

$$\beta = \widetilde{c}_1\beta_1 + \widetilde{c}_2\beta_2$$

and we can set $\delta := \frac{\dot{\beta}}{\dot{\gamma}_2}$.

Thus for $b = 0$ all geodesics are complete.

Case 2 $b \neq 0$ If we find curves α and η satisfying

$$\begin{aligned} (I)\ \dot{\alpha} &= \dot{\gamma}_2\eta \\ \text{and } (II)\ \dot{\eta} &= -\dot{\gamma}_2\alpha - \dot{\gamma}_1\eta \end{aligned}$$

the other equations of $(*)$ are satisfied for $b \neq 0$ by setting $\beta = \frac{1}{b}(\dot{\gamma}_1 - a\alpha)$ and $\delta := \frac{1}{b}(\dot{\gamma}_2 - a\eta)$. We also have to ensure that the determinant of g does not vanish.

$$\begin{aligned} det(g) &= \alpha\delta - \beta\eta \\ &= \alpha\tfrac{1}{b}(\dot{\gamma}_2 - a\eta) - \tfrac{1}{b}(\dot{\gamma}_1 - a\alpha)\eta \\ &= \tfrac{1}{b}(\alpha\dot{\gamma}_2 - \eta\dot{\gamma}_1) \\ &\overset{!}{\neq} 0 \end{aligned}$$

Consequently although $\alpha = const\cdot\dot{\gamma}_1$ and $\eta = const\cdot\dot{\gamma}_2$ is a solution of the differential equations this is no solution of our problem, since it is outside $Gl(2)$.

Case 2.1 $\dot{\gamma}_1 = \frac{c^2}{2}$ and $\dot{\gamma}_2 = 0$

The equations to be solved are

$$\begin{aligned} (I)\ \dot{\alpha} &= 0 \\ \text{and } (II)\ \dot{\eta} &= -\tfrac{c^2}{2}\eta. \end{aligned}$$

Consequently we get $\alpha = const$ and $\eta = d\exp(-\frac{c^2}{2}t+e)$. We obtain for β and δ: $\beta = \frac{1}{b}(\frac{c^2}{2} - a\alpha)$ and $\delta := -\frac{a}{b}d\exp(-\frac{c^2}{2}t+e)$.

Case 2.2 $\dot{\gamma}_1 = \frac{c}{\sqrt{2}}\tanh(\sqrt{2}ct+d)$ and $\dot{\gamma}_2 = \pm\frac{c}{\cosh(\sqrt{2}ct+d)}$

We combine the differential equations (I) and (II) to find one equation for α.

$$\begin{aligned} \dot{\eta} &\overset{(I)}{=} \frac{d}{dt}\left(\frac{\dot{\alpha}}{\dot{\gamma}_2}\right) \\ &= \frac{\ddot{\alpha}}{\dot{\gamma}_2} - \frac{\dot{\alpha}}{\dot{\gamma}_2^2}\underbrace{\ddot{\gamma}_2}_{=-2\dot{\gamma}_1\dot{\gamma}_2} \\ \dot{\eta} &\overset{(II)}{=} \alpha\dot{\gamma}_2 - \eta\dot{\gamma}_1 \\ &\overset{(I)}{=} \alpha\dot{\gamma}_2 - \frac{\dot{\alpha}\dot{\gamma}_1}{\dot{\gamma}_2} \end{aligned}$$

Hence we have to solve the following equation for α:

$$\ddot{\alpha} + 3\dot{\gamma}_1\dot{\alpha} + \dot{\gamma}_2^2\alpha = 0.$$

This is exactly the same equation we already solved in case 1.2. I.e. the curves α and $\eta = \frac{\dot{\alpha}}{\dot{\gamma}_2}$ are defined on $\mathbb{R}$ and so are β and δ.

For all $a, b \in \mathbb{R}$ we have found all curves $\gamma : \mathbb{R} \longrightarrow M$ and $g : \mathbb{R} \longrightarrow Gl(2)$ with $\omega\left(\frac{d}{dt}R_g \circ \sigma \circ \gamma\right) = \begin{pmatrix} a \\ b \end{pmatrix} \oplus 0.$

So the geodesics of the Cartan geometry $(\mathcal{G}, \pi, M; \omega)$ are

$$\begin{aligned} \text{Case 1 } \gamma(t) &= \begin{pmatrix} ht + j \\ k \end{pmatrix}, \\ \text{Case 2 } \gamma(t) &= \begin{pmatrix} \frac{1}{2}\log\left(\cosh(\sqrt{2}ct + d)\right) + e \\ \pm\sqrt{2}\arctan\left(\sinh(d) + \cosh(d)\tanh\left(\frac{ct}{\sqrt{2}}\right)\right) + f \end{pmatrix}, \end{aligned}$$

with $c, d, \ldots, k \in \mathbb{R}$ constant. We have found especially that all geodesics are defined on $\mathbb{R}$. Therefore, $(\mathcal{G}, \pi, M; \omega)$ is geodesically complete.

We now set

$$\gamma(t) := \begin{pmatrix} 3\log t \\ \sqrt{2}\log t \end{pmatrix} \text{ and } g(t) := \begin{pmatrix} \frac{3}{at} & -\frac{5}{aet^2} \\ \frac{\sqrt{2}}{at} & \frac{5\sqrt{2}}{aet^2} \end{pmatrix}.$$

Hence we have $g^{-1} = \begin{pmatrix} \frac{at}{4} & \frac{at}{4\sqrt{2}} \\ -\frac{aet^2}{20} & \frac{3aet^2}{20\sqrt{2}} \end{pmatrix}$ and therefore

$$\begin{aligned} \omega\left(\tfrac{d}{dt}R_g \circ \sigma \circ \gamma\right) &= g^{-1}\begin{pmatrix} \frac{3}{t} \\ \frac{\sqrt{2}}{t} \end{pmatrix} + g^{-1}\begin{pmatrix} 0 & -\frac{\sqrt{2}}{t} \\ \frac{\sqrt{2}}{t} & \frac{3}{t} \end{pmatrix} g + g^{-1}\dot{g} \\ &= \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{a}{4} & \frac{a}{4\sqrt{2}} \\ \frac{3aet}{20} & \frac{11aet}{20\sqrt{2}} \end{pmatrix} g + \begin{pmatrix} -\frac{1}{t} & 0 \\ 0 & -\frac{2}{t} \end{pmatrix} \\ &= \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix}. \end{aligned}$$

Thus γ is actually an ω-constant curve of finite length which cannot be prolonged for t approaching zero. Consequently the given Cartan geometry $(\mathcal{G}, \pi, M; \omega)$ is not ω-complete although it is geodesically complete.

We can therefore state in general

$$\begin{array}{c} \text{Cartan completeness} \\ \Downarrow \ \not\Uparrow \\ \omega\text{-completeness} \\ \Downarrow \ \not\Uparrow \\ \text{geodesical completeness} \end{array}.$$

6.4 Local Completeness

In Section 6.3 we discussed global concepts of completenes. Locally we can study the boundaries of neighbourhoods of a point and compare the Cartan boundary with the topological boundary defined by the embedding.

Definition 6.4 *A manifold M is said to be locally complete at a point $x \in M$ if there is a neighbourhood $U \subset M$ of x such that the Cartan boundary of U is equal to the topological boundary defined by the embedding $U \hookrightarrow M$, $\partial_{CB}U = \overline{U}^M \setminus U$.*
A manifold is said to be locally complete if it is locally complete at every point.

Lemma 6.5 *A Cartan complete manifold is locally complete.*

Proof: Let M be a Cartan complete manifold and $(\mathcal{G}, \pi, M; \omega)$ the Cartan bundle endowed with the Cartan connection ω. Let $U \subset M$ be an arbitrary open subset of M. Then $\mathcal{G}_U := \pi^{-1}(U) \xrightarrow{\pi} U$ is the Cartan bundle over U and $\omega|_{\mathcal{G}_U}$ is its Cartan connection. For any point $x \in \overline{U}^M \setminus U$ from the topological boundary of U we have a curve $\gamma : [0,1] \longrightarrow M$ with $\gamma\big([0,1)\big)$ being a subset of U and $\gamma(1) = x \in M$. Lifting this curve to a curve $\gamma^* : [0,1] \longrightarrow \mathcal{G}$ yields a curve of finite length in $\mathcal{G}$ and $\gamma^*|_{[0,1)}$ is a curve of finite length in $\mathcal{G}_U$ which is inextensible in $\mathcal{G}_U$. Hence x is also a point from the Cartan boundary of U, that is $\overline{U}^M \setminus U \subset \partial_{CB} U$. The other way round let x be a point in the Cartan boundary of U. This implies the existence of a curve $\gamma^* : [0,1) \longrightarrow \mathcal{G}_U$ of finite length which is inextensible in $\mathcal{G}_U$. However since $\mathcal{G}$ is Cartan complete, γ^* can be prolonged in $\mathcal{G}$ and its projection is a curve $\gamma : [0,1] \longrightarrow M$ with endpoint $x = \gamma(1) \in M$ and $\gamma|_{[0,1)}$ is a curve in U. Thus x is also a point in the topological boundary of U and we obtain $\partial_{CB} U \subset \overline{U}^M \setminus U$. As we have proven for a Cartan complete manifold the Cartan boundary of any open subset is identical to its topological boundary. So especially a Cartan complete manifold is also locally complete.

□

Lemma 6.6 *Let M be a locally complete manifold and $A \subset M$ be a closed subset of M. Then $M \setminus A$ is locally complete.*

Proof: Let $x \in M \setminus A$ be an arbitrary point. Since M is locally complete we have a neighbourhood $U \subset M$ of x whose Cartan boundary coincides with the topological boundary. We can furthermore find a neighbourhood $V \subset M \setminus A$ of x which is open in M and a subset of U. Analogously to the prove above the Cartan boundary of V is the same as its topological boundary, $\partial_{CB} V = \overline{V}^M \setminus V = \overline{V}^{M \setminus A} \setminus V$. Hence $M \setminus A$ is locally complete.

□

Please note, that a Cartan complete manifold will no longer be Cartan complete if a closed subset is being removed. Nevertheless it will still be locally complete.
Although there are no vertical curves defining boundary points there might be a non-vertical inextendable curve of finite length, whose projection lies within a compact subset of M, causing M to be not locally complete. We will now discuss an example from [Sch73] of a manifold which is at no point locally complete.
Let $M = \mathbb{R}^2$ be the two dimensional affine space and $(\mathcal{GL}(2), \pi, M; Gl(2))$ be the trivial $Gl(2)$-principal bundle with the global section

$$s : M \ni (x_1, x_2) \mapsto (e_1, e_2)_{(x_1,x_2)} \in \mathcal{GL}(2)_{(x_1,x_2)}.$$

Let A be the connection on $\mathcal{GL}(2)$ defined with the help of the section s by

$$\begin{aligned} A^s_{(x_1,x_2)}(e_1) &= (s^*A)_{(x_1,x_2)}(e_1) = \begin{pmatrix} x_2 & 0 \\ 0 & 0 \end{pmatrix} \text{ and} \\ A^s_{(x_1,x_2)}(e_2) &= (s^*A)_{(x_1,x_2)}(e_2) = \begin{pmatrix} 0 & 0 \\ 0 & -x_1 \end{pmatrix}. \end{aligned}$$

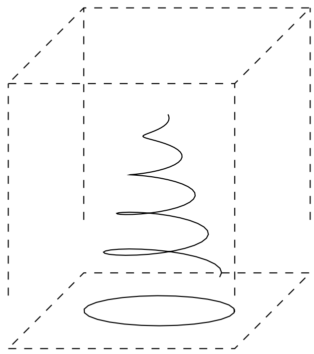
The corresponding connection $\nabla : \Gamma(TM) \longrightarrow \Gamma(TM^* \otimes TM)$ is given by

$$\nabla_{e_1} e_1 = x_2 e_1,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_2} e_1 = 0,\ \nabla_{e_2} e_2 = -x_1 e_2.$$

Please note that A is not a Levi Civita connection. Let us take a look at the curvature of A in order to determine the Lie algebra of the holonomy group of this connection.

$$\begin{aligned}
\mathcal{R}(e_1,e_2) = s^*\Omega^A(e_1,e_2) &= dA\big(ds(e_1), ds(e_2)\big) + \underbrace{\big[A(ds(e_1)), A(ds(e_2))\big]}_{=0} \\
&= ds(e_1)\big(A \circ ds(e_2)\big) - ds(e_2)\big(A \circ ds(e_1)\big) - s^*A\big(\underbrace{[e_1,e_2]}_{=0}\big) \\
&= ds(e_1)\begin{pmatrix} 0 & 0 \\ 0 & -x_1 \end{pmatrix} - ds(e_2)\begin{pmatrix} x_2 & 0 \\ 0 & 0 \end{pmatrix} \\
&= -Id
\end{aligned}$$

According to [KN63] the restricted holonomy group at a point x of a real ananlytic linear connection is completely determined by the values of all successive covariant differentials of the Riemannian curvature tensor $\nabla^k\mathcal{R}$ at the point x.
Hence $\mathfrak{hol}^A = span(Id) = \{a \cdot Id \mid a \in \mathbb{R}\}$.
So there are closed curves γ with horizontal lifts γ^* "spiraling up" as we go around γ again and again.



If the factor between two succeding loops of γ^* is greater than 1 the length of the loops of γ^* will decrese, since it is the length of the projection, that is to say of γ, measured in the frame γ^*.
Let us explicitly write down an example for such a curve and its horizontal lift.
A vector field $X = X_1e_1 + X_2e_2$ is parallel along a given curve $\gamma : I \longrightarrow M$ if and only if

$$0 = \frac{\nabla X}{dt} = \left(\dot{X}_1 + X_1\dot{\gamma}_1\gamma_2\right)e_1 + \left(\dot{X}_2 - X_2\dot{\gamma}_2\gamma_1\right)e_2.$$

Choosing $\gamma_1(t) := \cos(2\pi t)$ and $\gamma_2(t) := \sin(2\pi t)$ the vector field X with the coordinates $X_1 = a \cdot \underbrace{e^{2\pi\left(\frac{t}{2} - \frac{sin(4\pi t)}{8\pi}\right)}}_{:=\alpha(t)}$ and $X_2 = b \cdot \underbrace{e^{2\pi\left(\frac{t}{2} + \frac{sin(4\pi t)}{8\pi}\right)}}_{:=\beta(t)}$, with $a, b \in \mathbb{R}$ constant, is parallel along the curve γ. Hence a horizontal lift of γ is given by

$$\begin{aligned}
\gamma^*(t) &= (\alpha e_1 + \beta e_2, -\alpha e_1 + \beta e_2) \\
&= (e_1, e_2) \cdot \begin{pmatrix} \alpha(t) & -\alpha(t) \\ \beta(t) & \beta(t) \end{pmatrix} \\
&= (s \circ \gamma(t)) \cdot \begin{pmatrix} \alpha(t) & -\alpha(t) \\ \beta(t) & \beta(t) \end{pmatrix}.
\end{aligned}$$

Since the Cartan geometry of this example is modeled on a reductive space, the Cartan connection is given by $\omega := A + \theta$, where θ denotes the displacement form. And the length of γ^* is the length of γ measured in the parallely propagated frame γ^*. For the horizontal lift γ^* we have

$$\begin{aligned}
\omega\left(\dot{\gamma^*}(t)\right) &= \underbrace{A(\dot{\gamma^*}(t))}_{=0} + \theta(\dot{\gamma^*}(t)) \\
&= [\gamma^*(t)]^{-1}\big(d\pi \circ \dot{\gamma^*}(t)\big) \\
&= \left[(e_1, e_2) \cdot \begin{pmatrix} \alpha & -\alpha \\ \beta & \beta \end{pmatrix}\right]^{-1} \big(\dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2\big) \\
&= \tfrac{1}{2\alpha\beta} \cdot \begin{pmatrix} \beta & \alpha \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \end{pmatrix} \\
&= \tfrac{1}{2\alpha\beta} \cdot \begin{pmatrix} \beta\dot{\gamma}_1 + \alpha\dot{\gamma}_2 \\ -\beta\dot{\gamma}_1 + \alpha\dot{\gamma}_2 \end{pmatrix}.
\end{aligned}$$

Consequently the length of $\gamma^* : [0, \infty) \longrightarrow \mathcal{GL}(2)$ is finite:

$$\begin{aligned}
\ell(\gamma^*) &= \textstyle\int_0^\infty \left\|\omega(\dot{\gamma^*}(t))\right\| dt \\
&= \textstyle\int_0^\infty \frac{1}{2\alpha\beta} \sqrt{(\beta\dot{\gamma}_1 + \alpha\dot{\gamma}_2)^2 + (-\beta\dot{\gamma}_1 + \alpha\dot{\gamma}_2)^2} dt \\
&= \textstyle\int_0^\infty \frac{1}{2\alpha\beta} \sqrt{2\beta^2\dot{\gamma}_1^2 + 2\alpha^2\dot{\gamma}_2^2} dt \\
&= \textstyle\int_0^\infty \frac{1}{2} e^{-2\pi t} \sqrt{8\pi^2 e^{4\pi\left(\frac{t}{2} + \frac{sin(4\pi t)}{8\pi}\right)} \sin^2(2\pi t) + 8\pi^2 e^{4\pi\left(\frac{t}{2} - \frac{sin(4\pi t)}{8\pi}\right)} \cos^2(2\pi t)} dt \\
&= \textstyle\int_0^\infty \pi e^{-\pi t} \sqrt{2e^{\frac{sin(4\pi t)}{2}} \sin^2(2\pi t) + 2e^{-\frac{sin(4\pi t)}{2}} \cos^2(2\pi t)} dt \\
&\leq 2\pi e^{\frac{1}{4}} \textstyle\int_0^\infty e^{-\pi t} dt \\
&= 2\pi e^{\frac{1}{4}} \left[-\frac{1}{\pi} e^{-\pi t}\right]_0^\infty \\
&= 2e^{\frac{1}{4}}.
\end{aligned}$$

Since the curve γ can be rescaled and moved, we have for every point $x \in M = \mathbb{R}^2$ and every neighbourhood $U \subset M$ of x a curve which has an inextendable horizontal lift of finite length, i.e. the Cartan boundary of U contains more points than the boundary $\overline{U} \setminus U$ defined by the embedding $U \hookrightarrow M$. Hence (M, A) is at no point locally complete.

6.5 Degeneration of the Boundary Fibres in the Reductive Case

As we have seen, the group P acts continuously from the right on $\overline{\mathcal{G}}$. However the fibres of the boundary point may degenerate and hence $\overline{\mathcal{G}}$ may fail to be a principal bundle.
In this subsection, based on [Cla78] and [Cla79], we want to take a closer look at this degeneration in the case of Cartan geometries modeled on a reductive space.
We will use the same notations as in Subsection 6.2.3, that is G/P is again a reductive space, $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{m}$ with $\mathfrak{m}$ being $Ad(P)$-invariant. The Cartan connections splits into a principal bundle connection and a displacement form, $\omega = A + \theta$.
As we have seen in Subsection 6.2.3 for Cartan geometries modeled on a reductive space it suffices to consider the horizontal curves for determining the Cartan boundary. If we have a degenerated boundary fibre, we have an inextendable curve $\gamma : [0, 1) \longrightarrow \mathcal{G}$ of finite length and a $p \in P$ such that γ and $R_p \circ \gamma$ define the same boundary point. However this means that we further have a sequence $(t_n)_{n\in\mathbb{N}}$ tending to one and curves $\delta_n : [0, 1] \longrightarrow \mathcal{G}$ with $\delta(0) = \gamma(t_n)$, $\delta(1) = R_p \circ \gamma(t_n)$ and the length of δ_n tends to zero. Let δ_n^* be the horizontal curve starting in $\delta_n(0)$ with the same projection as δ_n. Then we can write for some curve $p_n : [0, 1] \longrightarrow P$ that $\delta_n = R_{p_n} \circ \delta_n^*$. As we have seen in Subsection 6.2.3 the length of the curve p_n is bounded by the length of δ_n and thus tends to zero for n going to infinity.

$$
\begin{aligned}
\ell(\delta_n) &= \int_0^1 \left\|\omega\left(\tfrac{d}{dt}(R_{p_n}\circ\delta_n^*)\right)\right\| dt \\
&= \int_0^1 \Big(\| \underbrace{Ad(p_n^{-1})\circ\omega(\delta_n^*)}_{\in\mathfrak{m}} \| + \| \underbrace{dL_{p_n^{-1}}\widetilde{\dot{p}_n}}_{\in\mathfrak{p}} \|\Big) dt \\
&= \int_0^1 \left\|Ad(p_n^{-1})\circ\omega(\delta_n^*)\right\| dt + \ell(p_n)
\end{aligned}
$$

With the length of p_n tending to zero we have $p_n(1) \overset{n\to\infty}{\longrightarrow} e$ and therefore

$$\delta_n^*(1) \overset{n\to\infty}{\longrightarrow} \delta_n(1) = R_p(\gamma(t_n)).$$

So in order to see, if a boundary fibre degenerates, we should take a look at horizontal curves with starting and endpoint in the same fibre, that is to say the projection would be a loop. If a boundary fibre degenerates there should be a sequence of such curves, such that the starting points tend to the boundary and the curves become arbitrarily short. Hence holonomy accessible along short curves offers to be the tool to describe these ideas precisely. With $\Omega(x) := \{\gamma : [0,1] \longrightarrow M \mid \gamma(0) = x = \gamma(1), \gamma \text{ piecewise smooth}\}$ we denote the space of all loops at the point $x \in M$, and γ_v^* is the horizontal lift of γ with starting point $\gamma_v^*(0) = v$. For a point $v \in \mathcal{G}$ and a number $k \in \mathbb{R}_+$ we define

$$Hol^k(v) := \{p \in P \mid \exists\gamma \in \Omega(\pi(v)) : \gamma_v^*(1) = R_p \circ \gamma_v^*(0), \ell(\gamma_v^*) \le k\}$$

to be the part of the holonomy group accessible along curves of length smaller or equal to $k \in \mathbb{R}_+$. Note that this construction depends on having a bundle connection, defining the horizontal lifts of curves.

We now want to define the singular holonomy group $G_{\overline{x}}(\kappa)$. Let $\kappa : [0,1) \longrightarrow \mathcal{G}$ be an inextensible, horizontal curve of finite length, i.e. κ defines a boundary point $\overline{u} \in \overline{\mathcal{G}}$. Denote with $\overline{x} := \pi(\overline{u})$ the corresponding point of the Cartan boundary $\partial_{CB}M$. We set

$$
\begin{aligned}
G_{\overline{x}}(\kappa) &:= \textstyle\bigcap_{k\in\mathbb{R}_+} \overline{\bigcup_{t\in[0,1)} Hol^k(\kappa(t))} \qquad \text{(with the bar } \bar{\cdot} \text{ we denote the closure in } P) \\
&= \left(\textstyle\bigcap_{k\in\mathbb{R}_+} cl \left\{ p \in P \;\middle|\; \begin{array}{c} \exists\gamma \in \Omega(\pi(u)) : \ell(\gamma_u^*) \le k, \gamma_u^*(1) = R_p u, \\ \text{with } u = \kappa(t) \text{ for a } t \in [0,1) \end{array} \right\} \right).
\end{aligned}
$$

Within the manifold the singular holonomy group is trivial. More precisely

Lemma 6.7 *For any point $x \in M$ and any horizontal curve $\kappa : [0,1] \longrightarrow \mathcal{G}$ ending over x, $\pi \circ \kappa(1) = x$, the singular holonomy group is trivial, $G_x(\kappa) = \{e\}$.*

Proof: For any fixed point u in the Riemannian manifold $\mathcal{G}$ any sequence of curves $\gamma_n : [0,1] \longrightarrow \mathcal{G}$, starting at u with decreasing length $\ell(\gamma_n) \overset{n\to\infty}{\longrightarrow} 0$, converges to the stationary curve $\gamma_\infty \equiv u$. So for all $u \in \mathcal{G}$ we have $Hol^k(u) \overset{k\to 0}{\longrightarrow} \{e\}$. For $u \in \mathcal{G}$ we have a neighbourhood U containing u such that the closure of U is a compact subset of $\mathcal{G}$. Hence within $\overline{U}$ the convergence $Hol^k(\cdot) \overset{k\to 0}{\longrightarrow} \{e\}$ is uniform.

With the curve κ being defined on the compact intervall $[0,1]$ we can cover the whole image of κ by a finite number of such neighbourhoods. This gives the uniform convergence of $Hol^k(\kappa(t)) \overset{k\to 0}{\longrightarrow} \{e\}$ on the whole interval $t \in [0,1]$ and thus proves that the singular holonomy group is trivial for any point $x \in M$, $G_x(\kappa) = \bigcap_{k\in\mathbb{R}_+} \overline{\bigcup_{t\in[0,1)} Hol^k(\kappa(t))} = \{e\}$.

$\square$

A consequence from the lemma above is that the singular holonomy group does not depend on parts of κ which are "far away" from the boundary.

Corollary 6.1 *Let $\overline{x} \in \partial_{CB} M$ be a point of the Cartan boundary of M and $\kappa : [0,1) \longrightarrow \mathcal{G}$ a horizontal, inextensible curve of finite length ending over $\overline{x}$, that is $\lim_{t \to 1} \pi \circ \kappa(t) = \overline{x}$. Then we have for the singular holonomy group*

$$G_{\overline{x}}(\kappa) = G_{\overline{x}}(\kappa|_{[\lambda,1)}) \text{ for all } \lambda \in [0,1).$$

Proof: We have

$$\begin{aligned} G_{\overline{x}}(\kappa) &= \bigcap_{k \in \mathbb{R}_+} \overline{\bigcup_{t \in [0,1)} Hol^k(\kappa(t))} \\ &= \bigcap_{k \in \mathbb{R}_+} \left(\overline{\bigcup_{t \in [0,\lambda)} Hol^k(\kappa(t))} \cup \overline{\bigcup_{t \in [\lambda,1)} Hol^k(\kappa(t))} \right). \end{aligned}$$

Since the holonomy group at a point $\kappa(t)$ accessible along shorter curves is contained in the holonomy group accessible along longer curves, that is to say $Hol^{k_1}(\kappa(t)) \subset Hol^{k_2}(\kappa(t))$ for all $0 < k_1 < k_2$, we can write for every $\lambda \in [0,1)$

$$\begin{aligned} G_{\overline{x}}(\kappa) &= \left(\bigcap_{k \in \mathbb{R}_+} \overline{\bigcup_{t \in [0,\lambda)} Hol^k(\kappa(t))} \right) \cup \left(\bigcap_{k \in \mathbb{R}_+} \overline{\bigcup_{t \in [\lambda,1)} Hol^k(\kappa(t))} \right) \\ &= G_{\kappa(\lambda)}(\kappa|_{[0,\lambda)}) \cup G_{\overline{x}}(\kappa|_{[\lambda,1)}). \end{aligned}$$

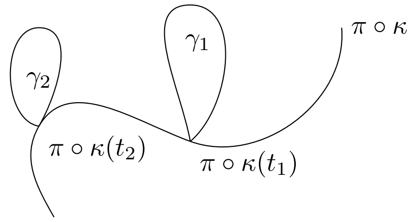
However with $G_{\kappa(\lambda)}(\kappa|_{[0,\lambda)})$ being trivial this is the statement claimed. The singular holonomy group does not depend on parts of κ which are not close to the boundary:

$$G_{\overline{x}}(\kappa) = G_{\overline{x}}(\kappa|_{[\lambda,1]}) \text{ for all } \lambda \in [0,1).$$

$\square$

Lemma 6.8 *$G_{\overline{x}}(\kappa)$ is a closed subgroup of P.*

Proof: We have to prove that $G_{\overline{x}}(\kappa)$ is closed under multiplication. Let $p_1, p_2 \in G_{\overline{x}}(\kappa)$ be fix. For any $a \in \mathbb{R}_+$ we can choose a $\lambda \in [0,1)$ such that $\ell(\kappa|_{[\lambda,1)}) \leq \min \left\{ \frac{a}{4\|p_1^{-1}\|}, \frac{a}{4\|p_2^{-1}p_1^{-1}\|} \right\}$. Here $\|\cdot\|$ denotes the operator norm given by the euclidian product, $\|A\| := \sup_{v \neq 0} \frac{\|Av\|}{\|v\|}$. With the corollary above we know $p_1, p_2 \in G_{\overline{x}}(\kappa|_{[\lambda,1)})$. Hence we have two short enough loops γ_1 and γ_2 in M generating p_1 and p_2, $\gamma_{1/2}^*(1) = R_{p_{1/2}} \gamma_{1/2}^*(0)$. More precisely we have $\gamma_1 \in \Omega(\pi \circ \kappa(t_1))$ and $\gamma_2 \in \Omega(\pi \circ \kappa(t_2))$ with $t_1, t_2 \in [\lambda, 1)$ satisfying $\ell(\gamma_1^*) \leq \frac{a}{4}$ and $\ell(\gamma_2^*) \leq \frac{a}{4\|p_1^{-1}\|}$ and generating p_1 respectively p_2.



Now we define a new loop in M by first going along the path of γ_1 then going along $\pi \circ \kappa$ from $\pi \circ \kappa(t_1)$ to $\pi \circ \kappa(t_2)$, next going along γ_2 and finally going back to the starting point $\pi \circ \kappa(t_1)$ along $\pi \circ \kappa$:

$$\begin{aligned} \gamma &:= \gamma_1 * (\pi \circ \kappa)|_{[t_1,t_2]} * \gamma_2 * (\pi \circ \kappa)|_{[t_2,t_1]}, \\ \text{i.e. } \gamma(t) &:= \begin{cases} \gamma_1(4t) & , 0 \leq t \leq \frac{1}{4} \\ \pi \circ \kappa\big(t_1 + (t_2 - t_1)(4t - 1)\big) & , \frac{1}{4} \leq t \leq \frac{1}{2} \\ \gamma_2(4t - 2) & , \frac{1}{2} \leq t \leq \frac{3}{4} \\ \pi \circ \kappa\big(t_2 + (t_1 - t_2)(4t - 3)\big) & , \frac{3}{4} \leq t \leq 1 \end{cases}. \end{aligned}$$

In the reductive case the horizontal subspaces are P-invariant, so the horizontal lift of γ is

$$\gamma^* = \gamma_1^* * R_{p_1}\kappa|_{[t_1,t_2]} * R_{p_1}\gamma_2^* * R_{p_1p_2}\kappa|_{[t_2,t_1]}$$

and we have $\gamma^*(1) = R_{p_1p_2}\kappa(t_1) = R_{p_1p_2}\gamma^*(0)$. The length of the horizontal lift of γ satisfies

$$\begin{aligned}\ell(\gamma^*) &\le \ell\big(\gamma_1^*\big) + \ell\big(R_{p_1}\kappa|_{[\lambda,1)}\big) + \ell\big(R_{p_1}\gamma_2^*\big) + \ell\big(R_{p_1p_2}\kappa|_{[\lambda,1)}\big)\\ &\le \ell\big(\gamma_1^*\big) + \|p_1^{-1}\|\ell\big(\kappa|_{[\lambda,1)}\big) + \|p_1^{-1}\|\ell\big(\gamma_2^*\big) + \|p_2^{-1}p_1^{-1}\|\ell\big(\kappa|_{[\lambda,1)}\big)\\ &\le a.\end{aligned}$$

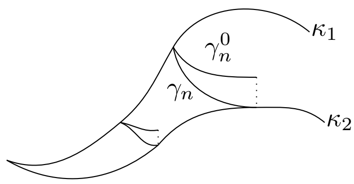
Since such loops $\gamma_{1/2}$ can be found for every number $a \in \mathbb{R}_+$ we obtain the result wanted, $p_1p_2 \in G_{\overline{x}}(\kappa)$. Hence $G_{\overline{x}}(\kappa) = \bigcap_{k\in\mathbb{R}_+} \overline{\bigcup_{t\in[0,1)} Hol^k(\kappa(t))}$ is a subgroup of P and it is also closed since the intersection of closed sets is closed.

□

We are now going to study in which way the singular holonomy group depends on the chosen horizontal curve κ.

Lemma 6.9 *Let κ_1 and κ_2 be two horizontal, inextendable curves in $\mathcal{G}$ of finite length ending over the same point $\overline{x}$, that is $\pi(\lim_{t\to 1}\kappa_1(t)) = \pi(\lim_{t\to 1}\kappa_2(t)) = \overline{x} \in \overline{M}$. Then $G_{\overline{x}}(\kappa_1)$ and $G_{\overline{x}}(\kappa_2)$ are conjugate in P.*

Proof: At first let us consider two curves κ_1 and κ_2 in $\mathcal{G}$ with the same limit in $\overline{\mathcal{G}}$, that is to say $\lim_{t\to 1}\kappa_1(t) = \lim_{t\to 1}\kappa_2(t) = \overline{u} \in \overline{\mathcal{G}}$ and $\pi(\overline{u}) = \overline{x}$. So for every sequence (t_n) with $t_n \overset{n\to\infty}{\longrightarrow} 1$ the distance between the corresponding curve points tends to zero, $d_\varrho(\kappa_1(t_n), \kappa_2(t_n)) \overset{n\to\infty}{\longrightarrow} 0$. Hence we have curves $\gamma_n : [0,1] \longrightarrow \mathcal{G}$ connecting $\kappa_1(t_n)$ with $\kappa_2(t_n)$ with decreasing length, $\ell(\gamma_n) \overset{n\to\infty}{\longrightarrow} 0$. Denote with γ_n^0 the horizontal lift of $\pi \circ \gamma_n$ with the same startig point $\gamma_n^0(0) = \gamma_n(0)$.



So for every $n \in \mathbb{N}$ we have a curve $p_n : [0,1] \longrightarrow P$ with $\gamma_n = R_{p_n} \circ \gamma_n^0$. Note that $p_n(0) = Id$. Recall that in the reductive case the Cartan connection consists of a displacementform and a bundle connection. We can write for the length of γ_n:

$$\begin{aligned}\ell(\gamma_n) &= \ell(R_{p_n} \circ \gamma_n^0)\\ &= \int_0^1 \big\|(\theta + A)\left(\tfrac{d}{dt}R_{p_n(t)} \circ \gamma_n^0(t)\right)\big\|\, dt\\ &= \int_0^1 \left(\big\|Ad(p_n^{-1}) \circ \theta(\dot\gamma_n^0)\big\| + \left\|dL_{p_n^{-1}} \circ \dot{p_n}\right\|\right) dt.\end{aligned}$$

Since the limit of the lengths of the curves γ_n is zero we also have for the lengths of the curves p_n that $\lim_{n\to\infty}\int_0^1 \|dL_{p_n^{-1}}\dot{p_n}\|dt = \lim_{n\to\infty}\ell(p_n) = 0$ and hence

$$p_n(t) \overset{n\to\infty}{\longrightarrow} p_n(0) = Id.$$

However this implies also that the lengths of the horizontal curves γ_n^0 tend to zero,

$$\ell(\gamma_n^0) = \int_0^1 \big\|\theta(\dot\gamma_n^0)\big\|\, dt \overset{n\to\infty}{\longrightarrow} 0.$$

Given $p_0 \in G_{\overline{x}}(\kappa_2)$ we know that for every $\lambda \in [0,1)$ p_0 is also an element of $G_{\overline{x}}(\kappa_2|_{[\lambda,1)})$. So we can find a sequence (t_n) with $t_n \in [1-\frac{1}{n}, 1)$ and loops $\hat{\gamma}_n \in \Omega(\pi \circ \kappa_2(t_n))$ generating p_0 with length of the horizontal lift smaller than $\frac{1}{n}$, i.e. $\ell(\hat{\gamma}_n^*) < \frac{1}{n}$ and $\hat{\gamma}_n^*(1) = R_{p_0} \circ \hat{\gamma}_n^*(0)$. Then $\breve{\gamma}_n := (\pi \circ \gamma_n^0) * \hat{\gamma}_n * (\pi \circ \gamma_n^0)^-$, where $^-$ means that the curve is run trough backwards, is a closed curve with starting and end point $\kappa_1(t_n)$. Its horizontal lift with the starting point $\breve{\gamma}_n^*(0) = \kappa_1(t_n)$ is $\breve{\gamma}_n^* = \gamma_n^0 * (R_{p_n(1)^{-1}} \circ \hat{\gamma}_n^*) * (R_{p_n(1)^{-1}} \circ R_{p_0} \circ R_{p_n(1)} \circ \gamma_n^0)^-$ with the end point $\breve{\gamma}_n^*(1) = R_{p_n(1)^{-1}} \circ R_{p_0} \circ R_{p_n(1)} \circ \kappa_1(t_n)$. Now with $p_n \overset{n\to\infty}{\longrightarrow} Id$, the lengths of γ_n^0 and $\hat{\gamma}_n^*$ tending to zero and p_0 being fix we obtain

$$\ell(\breve{\gamma}_n^*) \overset{n\to\infty}{\longrightarrow} 0 \text{ and } p_n(1) \cdot p_0 \cdot p_n(1)^{-1} \overset{n\to\infty}{\longrightarrow} p_0.$$

Hence we have $p_0 \in G_{\overline{x}}(\kappa_1)$ and thus we have for curves κ_1, κ_2 with the same limit $\overline{u}$ in $\overline{\mathcal{G}}$

$$G_{\overline{x}}(\kappa_1) = G_{\overline{x}}(\kappa_2).$$

Now let κ_1 and κ_2 be two curves in $\mathcal{G}$ ending over the same point $\overline{x} \in \overline{M}$. Then we have a $p \in P$ with $\lim_{t\to 1} R_p \circ \kappa_1(t) = \lim_{t \to 1} \kappa_2(t) \in \overline{\mathcal{G}}$. According to the argumentation above we have $G_{\overline{x}}(\kappa_2) = G_{\overline{x}}(R_p \circ \kappa_1)$ and hence $G_{\overline{x}}(\kappa_2) = L_{p^{-1}} \circ R_p \circ G_{\overline{x}}(\kappa_1)$.
So for two curves κ_1 and κ_2 in $\mathcal{G}$ ending over the same point $\overline{x} \in \overline{M}$, the groups $G_{\overline{x}}(\kappa_1)$ and $G_{\overline{x}}(\kappa_2)$ are conjugate in P.

$\square$

Proposition 6.3 *Let M be a manifold with the Cartan bundle $(\mathcal{G}, \pi, M; P)$ modelled on the reductive space $G/_P$ and endowed with the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Let $\overline{x} \in \partial_{CB}M$ be a point of the Cartan boundary of the manifold M. The fibre over $\overline{x}$ is isomorphic to the homogeneous space*

$$\pi^{-1}(\overline{x}) = P/_{\widetilde{P}}$$

where $\widetilde{P}$ is isomorphic to the singular holonomy group $G_{\overline{x}}(\kappa)$ defined by a horizontal inextendable curve $\kappa : [0,1) \longrightarrow \mathcal{G}$ of finite length approaching $\overline{x}$, $\pi\big(\lim_{t\to 1} \kappa(t)\big) = \overline{x}$.

Proof: Let $\overline{u} \in \pi^{-1}(\overline{x})$ be an element in the fibre over the boundary point $\overline{x} \in \partial_{CB}M$. Let furthermore $p \in G_{\overline{x}}(\kappa)$ be an element of the singular holonomy group with the inextensible horizontal curve $\kappa : [0,1) \longrightarrow \mathcal{G}$ defining $\overline{u}$, that is $\lim_{t\to 1} \kappa(t) = \overline{u}$. We can choose a sequence (t_n) in $[0,1)$ with $t_n \overset{n\to\infty}{\longrightarrow} 1$ and loops $\gamma_n \in \Omega(\pi \circ \kappa(t_n))$ with decreasing length of the horizontal lift $\ell(\gamma_n^*) \overset{n\to\infty}{\longrightarrow} 0$ and generating p, that is to say $\gamma_n^*(1) = R_p \circ \gamma_n^*(0)$. Hence the limits fulfill $\lim_{n\to\infty} \gamma_n^*(0) = \lim_{n\to\infty} \gamma_n^*(1) = \lim_{n\to\infty} R_p \circ \gamma_n^*(0)$. And with $\gamma_n^*(0) = \kappa(t_n) \overset{n\to\infty}{\longrightarrow} \overline{u}$ the identification $\overline{u} = R_p(\overline{u})$ follows.
The other way round let $\overline{u} \in \pi^{-1}(\overline{x})$ be an element in the fibre over $\overline{x} \in \partial_{CB}M$ and $p \in P$ with $\overline{u} = R_p(\overline{u})$. We have a horizontal curve $\kappa : [0,1) \longrightarrow \mathcal{G}$ defining $\overline{u}$, i.e. $\kappa(t) \overset{t\to 1}{\longrightarrow} \overline{u}$. Furthermore we have $d_\varrho(\kappa(t), R_p\kappa(t)) \overset{t\to 1}{\longrightarrow} 0$. Let (t_n) be a sequence in $[0,1)$ tending to 1. We have curves $\gamma_n : [0,1] \longrightarrow \mathcal{G}$ connecting $\kappa(t_n)$ with $R_p \circ \kappa(t_n)$ with decreasing length $\ell(\gamma_n) \overset{n\to\infty}{\longrightarrow} 0$. The projections of γ_n are loops in the base manifold M, $\pi \circ \gamma_n \in \Omega\big(\pi(\kappa(t_n))\big)$. We set $\gamma_n^0 := (\pi \circ \gamma_n)^*_{\gamma_n(0)}$ to be the horizontal lift of the loop $\pi \circ \gamma_n$ with the same starting point as γ_n. Following the arguments in the proof of the lemma above we obtain again for $\gamma_n = R_{p_n} \circ \gamma_n^0$

$$\ell(\gamma_n^0) \overset{n\to\infty}{\longrightarrow} 0 \text{ and } p_n(t) \overset{n\to\infty}{\longrightarrow} p_n(0) = Id.$$

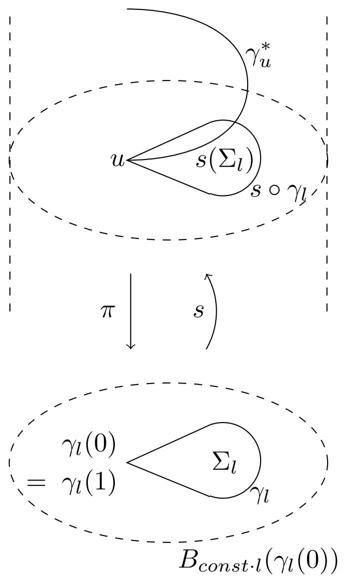
Hence γ_n^0 is the horizontal lift of the loop $\pi \circ \gamma_n \in \Omega(\pi \circ \kappa(t_n))$ with

$$\gamma_n^0(1) = R_{p_n(1)^{-1}} \circ \gamma_n(1) = R_{p_n(1)^{-1}} \circ R_p \circ \kappa(t_n) = \underbrace{R_{p_n(1)^{-1}}}_{\overset{n\to\infty}{\longrightarrow} Id} \circ R_p \circ \gamma_n^0(0).$$

Due to the singular holonomy group being closed we obtain $p \in G_{\overline{x}}(\kappa)$. Thus $G_{\overline{x}}(\kappa)$ is the isotropy group of $\overline{u}$.
Therefore, the fibre over a point of the Cartan boundary is isomorphic to $P/_{G_{\overline{x}}(\kappa)}$.

□

Using the path-ordered exponential and approximations of it (see [Baer09]) we obtain a relationship between the singular holonomy group and the curvature in the case of P being a group of matricies. More precisely let $\gamma_l : [0,1] \longrightarrow M$ be a small loop in the base manifold M, that is with respect to some metric the length of γ_l is $O(l)$ and the surface Σ_l bounded by γ_l is contained in the ball arround $\gamma_l(0)$ with radius $const \cdot l$. Its area is $area(\Sigma_l) = O(l^2)$. Furthermore let l be sufficiently small such that γ_l is contained in an open neighbourhood U of $\gamma_l(0)$ with a local section $s : U \longrightarrow \mathcal{G}_U$. Then we can actually chose the metric on the neighbourhood U by pulling back the metic on $\mathcal{G}$ with the section s.



The horizontal lift of γ_l with starting point $u \in \mathcal{G}_{\gamma(0)}$ can now be described via

$$\begin{aligned} \gamma_u^*(t) &= R_{p(t)} \circ s \circ \gamma(t) \\ \text{with } 0 &\overset{!}{=} A(\dot{\gamma_u^*}(t)) \\ &= Ad(p^{-1}(t)) \circ s^* A\big(\dot{\gamma}(t)\big) + dL_{p^{-1}(t)} \dot{p}(t). \end{aligned}$$

Thus we need to solve the differential equation $\dot{p}(t) = -dR_{p(t)} \circ \underbrace{s^* A\big(\dot{\gamma}(t)\big)}_{=:\Gamma(t)}$. However this equation is solved by the path-ordered exponential (see [Baer09])

$$\begin{aligned} p(t) &= Pexp\Big(-\int_0^t \Gamma(\tau) d\tau\Big) \\ &:= \sum_{j=0}^{\infty} (-1)^j \int_0^t \big(\int_0^{\tau_j} \dots \big(\int_0^{\tau_2} \Gamma(\tau_j) \cdots \Gamma(\tau_1) d\tau_1\big) \cdots d\tau_{j-1}\big) d\tau_j \\ &= lim_{N\to\infty} \big(Id - \tfrac{t}{N}\Gamma(\tfrac{N-1}{N}t)\big) \cdots \big(Id - \tfrac{t}{N}\Gamma(\tfrac{1}{N}t)\big) \cdot \big(Id - \tfrac{t}{N}\Gamma(0)\big). \end{aligned}$$

And according to [Baer09] this can be approximated by

$$p(1) = Id - \int_{\Sigma_l} s^* \Omega^A + O(l^3).$$

This reasons the conclusion of Clarke [Cla78] that "near a curvature singularity one would expect the boundary fibre to degenerate".
In the case of P being an abelian group of matricies calculations are a lot esier. Then the differential equation $\dot{p}(t) = -dR_{p(t)} \circ s^* A\big(\dot{\gamma}(t)\big)$ is solved by $p(t) = \exp\Big(-\int_0^t s^* A\big(\dot{\gamma}(\tau)\big) d\tau\Big)$. And using the theorem of Stokes we obtain

$$\begin{array}{rcl} p & = & p(1) \\ & = & \exp\Big(-\int_0^1 s^* A\big(\dot{\gamma}(\tau)\big) d\tau\Big) \\ & = & \exp\Big(-\int_{s^*\gamma} A\Big) \\ & \overset{\text{Stokes}}{=} & \exp\left(-\int_{s^*\Sigma} dA\right) \\ & = & \exp\left(-\int_{s^*\Sigma} \Omega^A\right). \end{array}$$

This yields again that "at a curvature singularity one would expect the boundary fibre to degenerate".

6.6 Future and Past Infinity Meet

Let us take a look at the manifold $M := (0, 2\pi) \times \mathbb{R}$ equipped with the Lorentzian metric $g = \sin^2\left(\frac{x_1}{2}\right) \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The Cartan bundle is the bundle of all orthogonal frames $\mathcal{O}(M, g)$ with $P = O(1, 1)$ acting from the right. In [Bos79] B.Bosshard showed that for this manifold past infinity and future infinity are identified with each other in the Cartan boundary. Furthermore this is an example of a manifold, where a boundary fibre degenerates.
At first we are going to determine the horizontal curves in the Cartan bundle $\mathcal{O}(M, g)$ equipped with the Levi Civita connection plus the displacement form as the Cartan connection, $\omega = A^{LC} + \theta$. The Christoffel symbols for the Lorentzian metric g with respect to the standard basis (e_1, e_2) are:

$$\begin{array}{l} \Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = \frac{1}{2} \cot \frac{x_1}{2} \text{ and} \\ \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{22}^2 = 0. \end{array}$$

Note that the Riemannian curvature tensor becomes singular for x_1 approaching 0 or 2π. More precisely we have

$$\mathcal{R}(e_1, e_2)_{(x_1, x_2)} = -\frac{1}{4 \sin^2 \frac{x_1}{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the sectional curvature is

$$K_{span\{e_1, e_2\}}(x_1, x_2) = \frac{1}{4 \sin^4 \frac{x_1}{2}}.$$

Thus the integral $\int_{\Sigma_l} s^* \Omega^{A^{LC}}$ contributing primarily to the singular holonomy group as discussed in Section 6.5 can take any values however small l is by choosing x_1 close enough to 0 or 2π. Consequently according to Section 6.5 this example is a good candidate for a degenerated boundary fibre.
We have a global section

$$\begin{array}{rcl} E : M & \longrightarrow & \mathcal{O}(M, g) \\ (x_1, x_2) & \mapsto & (E_1, E_2)_{(x_1, x_2)} \\ & := & \big(\sin^{-1}(\frac{x_1}{2}) e_1, \sin^{-1}(\frac{x_1}{2}) e_2\big)_{(x_1, x_2)}. \end{array}$$

Applying the Levi Civita connection A^{LC} to a vector $dE(\xi) = dE(\xi_1 e_1 + \xi_2 e_2)$, $\xi \in \mathfrak{X}(M)$ gives

$$\begin{aligned}
E^* A^{LC}(\xi)_{(x_1,x_2)} &= -g\left(\nabla^{LC}_{\xi}(E_1), E_2\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= -\sin^{-2}(\tfrac{x_1}{2}) g\left(\nabla^{LC}_{\xi} e_1, e_2\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= -\sin^{-2}(\tfrac{x_1}{2}) g\left((\xi_1 \Gamma^2_{11} + \xi_2 \Gamma^2_{21}) e_2, e_2\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \left(\xi_1 \Gamma^2_{11} + \xi_2 \Gamma^2_{21}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \tfrac{1}{2}\cot(\tfrac{x_1}{2})\xi_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

Any curve $\gamma^* : I \longrightarrow \mathcal{O}(M,g)$ projecting onto the curve $\gamma = (\gamma_1, \gamma_2) = \pi \circ \gamma^*$ can be written as $\gamma^* = E(\gamma)\underbrace{\begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}}_{=:A(\alpha)\in O_c(1,1)}$ for some curve $\alpha : I \longrightarrow \mathbb{R}$.

The curve γ^* is horizontal, if and only if the Levi Civita connection applied to its tangent vector vanishes. I.e.

$$\begin{aligned}
0 &\overset{!}{=} A^{LC}(\dot{\gamma}^*) \\
&= A^{LC}\left(dR_A \circ dE \circ \dot{\gamma}\right) + dL_{A^{-1}}\dot{A} \\
&= Ad(A^{-1}) \circ E^* A^{LC}(\dot{\gamma}) + dL_{A^{-1}}\dot{A} \\
&= A^{-1} \cdot \tfrac{1}{2}\cot(\tfrac{\gamma_1}{2})\dot{\gamma}_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot A + A^{-1} \cdot \dot{A}.
\end{aligned}$$

Hence

$$\begin{aligned}
\dot{A} &= \dot{\alpha}\begin{pmatrix} \sinh\alpha & \cosh\alpha \\ \cosh\alpha & \sinh\alpha \end{pmatrix} \\
&\overset{!}{=} -\tfrac{1}{2}\cot(\tfrac{\gamma_1}{2})\dot{\gamma}_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot A \\
&= -\tfrac{1}{2}\cot(\tfrac{\gamma_1}{2})\dot{\gamma}_2 \begin{pmatrix} \sinh\alpha & \cosh\alpha \\ \cosh\alpha & \sinh\alpha \end{pmatrix}.
\end{aligned}$$

And we obtain $\dot{\alpha} = -\frac{1}{2}\cot(\frac{\gamma_1}{2})\dot{\gamma}_2$.
With the tools above we can now easily find horizontal curves with given projections.
Our aim is to prove that past infinity and future infinity are identified with each other in the Cartan boundary. We will split the proof into the following parts.

- Two curves $E(t, const_1)$ and $E(t, const_1 + f(t))$ with $f(t) \overset{t\to 0}{\longrightarrow} 0$ define the same boundary point in $\mathcal{O}(M,g)$ for $t \longrightarrow 0$.

- Two curves $E(t, const_1 + const_2 \tan\frac{t}{2})$ and $R_A \circ E(t, const_1)$ define the same boundary point in $\mathcal{O}(M,g)$.

  Together with the statement above we obtain, that $R_A \circ E(t, const_1)$ defines the same boundary point as $E(t, const_1)$, that is the fiber through $\lim_{t\to 0} E(t, const_1)$ in $\overline{\mathcal{O}(M,g)}$ is degenerated.

- In the same way the fiber in $\overline{\mathcal{O}(M,g)}$ through $\lim_{t\to 2\pi} E(t, const_1)$ is degenerated.

- For every $\varepsilon > 0$ we have a horizontal lift $R_A \circ E(t, const + t)$ of the light like geodesic $\gamma = (t, const + t)$ with length smaller than ε.

- Linking the connecting curves from the items above gives curves γ_δ connecting $E(\delta, const)$ with $E(2\pi - \delta, const)$ with $\ell(\gamma_\delta) \stackrel{\delta \to 0}{\longrightarrow} 0$. Hence future infinity and past infinity are identified with each other in the Cartan boundary.

We will take a look at certain curves in M and their horizontal lifts in order to use them for the proof.

$(t, const)$ We have $\alpha = const$. So the horizontal lifts of $(t, const)$ are

$$R_A \circ E(t, const) \text{ with } A = A(\alpha) = const.$$

Since these curves are horizontal we obtain for their lengths:

$$\begin{aligned}
\ell\big(R_A \circ E(t, const)\big) &= \int_0^{2\pi} \|\omega(dR_A \circ dE(e_1))\| dt \\
&= \int_0^{2\pi} \|\theta(dR_A \circ dE(e_1))\| dt \\
&= \int_0^{2\pi} \|[R_A \circ E \circ \gamma]^{-1}(e_1)\| dt \\
&= \int_0^{2\pi} \left\| \big(\sin^{-1} \tfrac{t}{2} A\big)^{-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| dt \\
&= \int_0^{2\pi} \left\| \sin \tfrac{t}{2} \begin{pmatrix} \cosh\alpha \\ \sinh\alpha \end{pmatrix} \right\| dt \\
&= \sqrt{\cosh^2\alpha + \sinh^2\alpha} \int_0^{2\pi} \sin \tfrac{t}{2} dt \\
&= 4\sqrt{\cosh^2\alpha + \sinh^2\alpha}.
\end{aligned}$$

So all curves $R_A \circ E(t, const)$ with $A = const$ are curves of finite length and they are inextendable for $t \longrightarrow 0$. Therefore, they define at least one boundary point. The question has to be answered, whether they define the same boundary point.

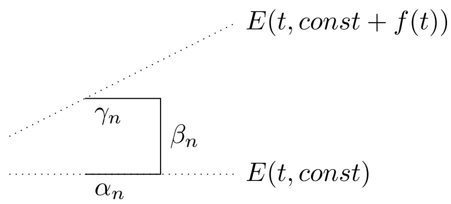
$(const, t)$ Setting $\alpha = \int \frac{1}{2} \cot \frac{const}{2} dt = a_1 t + a_2$ with $a_1, a_2 = const$ the horizontal lifts of the curves $(const, t)$ are:

$$R_{A(a_1 t + a_2)} \circ E(const, t) = R_{A(a_1 t) \cdot A(a_2)} \circ E(const, t) \text{ with } a_1, a_2 = const.$$

Now we will prove the facts needed.

- The sequences $E(t_n, const)$ and $E(t_n, const + f(t_n))$ with $f(t) \stackrel{t \to 0}{\longrightarrow} 0$ define the same boundary point in $\mathcal{O}(M, g)$ for $t_n \stackrel{n \to \infty}{\longrightarrow} 0$. Without loss of generality we assume f to be positive.

  We have already seen, that for $t_n \stackrel{n \to \infty}{\longrightarrow} 0$ the points $E(t_n, const)$ form a Cauchy sequence, that is to say they define a boundary point. We will prove that the distance between $E(t_n, const)$ and $E(t_n, const + f(t_n))$ approaches zero for t_n going to zero.



We set

$$\begin{aligned}
\alpha_n(t) &= E(t, const),\ t \in [t_n, f(t_n)^{\frac{1}{3}}] \text{ (or as the case may be } [f(t_n)^{\frac{1}{3}}, t_n]),\\
\beta_n(t) &= E\big(f(t_n)^{\frac{1}{3}}, const + tf(t_n)\big),\ t \in [0,1] \text{ and}\\
\gamma_n(t) &= E(t, const + f(t_n)),\ t \in [t_n, f(t_n)^{\frac{1}{3}}] \text{ (or as the case may be } [f(t_n)^{\frac{1}{3}}, t_n]).
\end{aligned}$$

Note that the curves α_n and γ_n are horizontal and β_n is not. Combined α_n, β_n and γ_n form a curve connecting $\alpha_n(t_n) = E(t_n, const)$ and $\gamma_n(t_n) = E(t_n, const + f(t_n))$. And we get the following bound for the distance between those points.

$$\begin{aligned}
& d_\rho\big(E(t_n, const), E(t_n, const + f(t_n))\big)\\
\leq\ & \ell(\alpha_n) + \ell(\beta_n) + \ell(\gamma_n)\\
=\ & 2\ell(\alpha_n) + \ell(\beta_n)\\
=\ & 2\left|\int_{t_n}^{f(t_n)^{\frac{1}{3}}} \sin\tfrac{t}{2} dt\right|\\
& + \int_0^1 \left(\left\| \left[E\big(f(t_n)^{\frac{1}{3}}, const + tf(t_n)\big)\right]^{-1} (f(t_n)e_2) \right\| + \|E^* A^{LC}(f(t_n)e_2)\| \right) dt\\
=\ & 4\left|\cos\tfrac{t_n}{2} - \cos\tfrac{f(t_n)^{\frac{1}{3}}}{2}\right| + \int_0^1 \left(\sin\tfrac{f(t_n)^{\frac{1}{3}}}{2} f(t_n) + \left\| \tfrac{1}{2}\cot\tfrac{f(t_n)^{\frac{1}{3}}}{2} f(t_n) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\| \right) dt\\
=\ & 4\left|\cos\tfrac{t_n}{2} - \cos\tfrac{f(t_n)^{\frac{1}{3}}}{2}\right| + \sin\tfrac{f(t_n)^{\frac{1}{3}}}{2} f(t_n) + \tfrac{1}{2}\cot\tfrac{f(t_n)^{\frac{1}{3}}}{2} f(t_n)\\
\overset{t_n \to 0}{\longrightarrow}\ & 0
\end{aligned}$$

Consequently the curves $E(t, const)$ and $E(t, const + f(t))$ with $f(t) \overset{t \to 0}{\longrightarrow} 0$ define the same boundary point in $\mathcal{O}(M, g)$ for t tending to zero.

- The sequenzes $E(t_n, c + 2a\tan\frac{t_n}{2})$ and $R_{A(a)} \circ E(t_n, c)$ define for $t_n \overset{n\to\infty}{\longrightarrow} 0$ the same boundary point in $\mathcal{O}(M, g)$. Here a and c are constants.

  We are looking for a horizontal lift δ of the curve $\big(t_n, c + (1-t)2a\tan\frac{t_n}{2}\big)$ with $t \in [0,1]$ which connects $E(t_n, c + 2a\tan\frac{t_n}{2})$ and $R_{A(a)} \circ E(t_n, c)$. We define δ to be to following curve:

$$\delta(t) \ = \ R_{A(at)} \circ E\left(t_n, c + (1-t)2a\tan\tfrac{t_n}{2}\right).$$

  The curve δ connects the two points

$$\begin{aligned}
\delta(0) &= E(t_n, c + 2a\tan\tfrac{t_n}{2}) \text{ and}\\
\delta(1) &= R_{A(a)} \circ E\left(t_n, c\right).
\end{aligned}$$

  The length of δ is:

$$\begin{aligned}
\ell(\delta|_{[0,1]}) &= \int_0^1 \left\|\omega(\dot{\delta}(t))\right\| dt\\
&= \int_0^1 \left\| \left[R_{A(at)} \circ E\left(t_n, c + (1-t)2a\tan\tfrac{t_n}{2}\right)\right]^{-1} \left(-2a\tan\tfrac{t_n}{2} e_2\right) \right\| dt\\
&= \int_0^1 \left\| \sin\tfrac{t_n}{2} A\left(-at\right)\left(-2a\tan\tfrac{t_n}{2} e_2\right) \right\| dt\\
&= \int_0^1 \left\| \sin\tfrac{t_n}{2} \begin{pmatrix} \cosh(-at) & \sinh(-at) \\ \sinh(-at) & \cosh(-at) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -2a\tan\frac{t_n}{2} \end{pmatrix} \right\| dt\\
&= \int_0^1 2a\tan\tfrac{t_n}{2}\sin\tfrac{t_n}{2}\sqrt{\sinh^2(at) + \cosh^2(at)dt}\\
&= \int_0^1 2a\tan\tfrac{t_n}{2}\sin\tfrac{t_n}{2}\sqrt{\cosh(2at)}dt\\
&\leq \int_0^1 2a\tan\tfrac{t_n}{2}\sin\tfrac{t_n}{2}\cosh(2at)dt\\
&= \tan\tfrac{t_n}{2}\sin\tfrac{t_n}{2}\left[\sinh(2at)\right]_0^1\\
&= \tan\tfrac{t_n}{2}\sin\tfrac{t_n}{2}\sinh 2a\\
&\overset{t_n \to 0}{\longrightarrow} 0.
\end{aligned}$$

We obtain the result wanted. The curves $E(t, c + 2a\tan\frac{t}{2})$ and $R_{A(a)} \circ E(t,c)$ define for $t \longrightarrow 0$ the same boundary point in $\mathcal{O}(M,g)$. Together with the first statement the degeneration of the fiber through $\lim_{t\to 0} E(t, const)$ in $\overline{\mathcal{O}(M,g)}$ is proven.

$$\pi^{-1} \circ \pi\big(\lim_{t\to 0} E(t, const)\big) = \{point\}$$

as was already clear from considering the curvature.

- In the same way the fiber in $\overline{\mathcal{O}(M,g)}$ through $\lim_{t\to 2\pi} E(t, const)$ is degenerated.

  This is true due to the symmetry of the Lorentzian metric $g_{(x_1,x_2)} = g_{(2\pi - x_1, x_2)}$. I.e. as in the case of t tending to zero the sequences $E(t_n, const)$ and $E(t_n, const + f(t_n))$ with $f(t) \overset{t\to 2\pi}{\longrightarrow} 0$ define the same boundary point in $\mathcal{O}(M,g)$ for $t_n \overset{n\to\infty}{\longrightarrow} 2\pi$. Furthermore the sequenzes $E(t_n, c - 2a\tan\frac{t_n}{2})$ and $R_{A(a)} \circ E(t_n, c)$ define for $t_n \overset{n\to\infty}{\longrightarrow} 2\pi$ the same boundary point in $\mathcal{O}(M,g)$. So the fiber through $\lim_{t\to 2\pi} E(t, const)$ in $\overline{\mathcal{O}(M,g)}$ is degenerated.

$$\pi^{-1} \circ \pi\big(\lim_{t\to 2\pi} E(t, const)\big) = \{point\}$$

- For every $\varepsilon > 0$ we have a horizontal lift γ_a of the light like geodesic $(t, const + t)$ with length smaller than ε.

  A horizontal lift of $(t, const + t)$ is for example

$$\gamma_a(t) = R_{A\left(\ln(\sin\frac{t}{2})\right)\cdot A(a)} \circ E(t, const + t).$$

Since γ_a is a horizontal lift the Cartan connection $\omega = \theta + A^g$ can be reduced to the displacement form θ and it is very easy to calculate the length of γ.

$$\begin{array}{rl}
 & \ell(\gamma_a) \\
= & \int_0^{2\pi} \|\theta(\dot{\gamma_a})\| dt \\
= & \int_0^{2\pi} \left\| \left[E(t, const+t)\cdot A\big(\ln(\sin\frac{t}{2})\big)\cdot A(a)\right]^{-1}(e_1+e_2)\right\| dt \\
= & \int_0^{2\pi} \left\| \sin\frac{t}{2} A(-a)\cdot A\big(-\ln(\sin\frac{t}{2})\big)\cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\| dt \\
= & \int_0^{2\pi} \left\| \sin\frac{t}{2} A(-a)\cdot \frac{1}{2}\begin{pmatrix} \sin\frac{t}{2} + \sin^{-1}\frac{t}{2} & -\sin\frac{t}{2} + \sin^{-1}\frac{t}{2} \\ -\sin\frac{t}{2} + \sin^{-1}\frac{t}{2} & \sin\frac{t}{2} + \sin^{-1}\frac{t}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\| dt \\
= & \int_0^{2\pi} \left\| \sin\frac{t}{2} \begin{pmatrix} \cosh a & -\sinh a \\ -\sinh a & \cosh a \end{pmatrix} \cdot \begin{pmatrix} \sin^{-1}\frac{t}{2} \\ \sin^{-1}\frac{t}{2} \end{pmatrix}\right\| dt \\
= & \int_0^{2\pi} \left\| \begin{pmatrix} \cosh a & -\sinh a \\ -\sinh a & \cosh a \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\| dt \\
= & \int_0^{2\pi} \sqrt{2(\cosh a - \sinh a)^2} dt \\
= & \int_0^{2\pi} \sqrt{2} e^{-a} dt \\
= & 2\pi\sqrt{2}e^{-a} \\
\overset{a\to\infty}{\longrightarrow} & 0
\end{array}$$

So however small we choose $\varepsilon > 0$, we have an $a \in \mathbb{R}$ such that γ_a has length smaller than ε.

- Linking the connecting curves from the items above gives curves γ_δ connecting $E(\delta, const)$ with $E(2\pi - \delta, const)$ with decreasing lengths $\ell(\gamma_\delta) \overset{\delta\to 0}{\longrightarrow} 0$. Hence future infinity and past infinity are identified with each other in the Cartan boundary.

Here are the parts of γ_δ in detail.

$$\begin{array}{cc}
\text{succeding points} & \text{length of the connecting curves} \\
E(\delta, const) & \\
\downarrow & \begin{array}{c} 4\left|\cos\frac{\delta}{2} - \cos\frac{(2a\tan\frac{\delta}{2})^{\frac{1}{3}}}{2}\right| \\ +2a\tan\frac{\delta}{2}\sin\frac{(2a\tan\frac{\delta}{2})^{\frac{1}{3}}}{2} \\ +a\tan\frac{\delta}{2}\cot\frac{(2a\tan\frac{\delta}{2})^{\frac{1}{3}}}{2} \end{array} \\
E(\delta, const + 2a\tan\frac{\delta}{2}) & \\
\downarrow & \tan\frac{\delta}{2}\sin\frac{\delta}{2}\sinh 2a \\
R_{A(a)} \circ E(\delta, const) &
\end{array}$$

Let us take a separate look at the horizontal lift of the light like geodesic. Starting with the point $R_{A(a)} \circ E(\delta, const) = \gamma_a(\delta)$ the horizontal curve

$$\begin{aligned}
\gamma_a(t) &= R_{A(a)} \circ R_{A\left(-\ln(\sin\frac{\delta}{2})\right)} \circ R_{A\left(\ln(\sin\frac{t}{2})\right)} \circ E(t, const - \delta + t) \\
&= R_{A\left(\ln(\sin\frac{t}{2})\right)\cdot A\left(a - \ln(\sin\frac{\delta}{2})\right)} \circ E(t, const - \delta + t)
\end{aligned}$$

leads to the point $\gamma_a(2\pi - \delta) = R_{A(a)} \circ E(2\pi - \delta, const + 2\pi - 2\delta)$.

The length of this curve is bounded by $\ell(\gamma_a|_{[\delta, 2\pi-\delta]}) < \ell(\gamma_a|_{[0,2\pi]}) = 2\pi\sqrt{2}e^{-a+\ln(\sin\frac{\delta}{2})}$.

Now we can continue connecting the points.

$$\begin{array}{cc}
\text{succeding points} & \text{length of the connecting curves} \\
R_{A(a)} \circ E(\delta, const) & \\
\downarrow & \ell < 2\pi\sqrt{2}e^{-a+\ln(\sin\frac{\delta}{2})} \\
R_{A(a)} \circ E(2\pi - \delta, const + 2\pi - 2\delta) & \\
\downarrow & \begin{array}{c} -\tan\frac{2\pi-\delta}{2}\sin\frac{2\pi-\delta}{2}\sinh 2a \\ = \tan\frac{\delta}{2}\sin\frac{\delta}{2}\sinh 2a \end{array} \\
E\left(2\pi - \delta, const + 2\pi - 2\delta + 2a\tan\frac{\delta}{2}\right) & \\
\downarrow & \begin{array}{c} 4\left|\cos\frac{2\pi-\delta}{2} - \cos\frac{2\pi-\left(2a\tan\frac{\delta}{2}\right)^{\frac{1}{3}}}{2}\right| \\ +2a\tan\frac{\delta}{2}\sin\frac{\left(2a\tan\frac{\delta}{2}\right)^{\frac{1}{3}}}{2} \\ +\frac{2a\tan\frac{\delta}{2}}{2}\cot\frac{\left(2a\tan\frac{\delta}{2}\right)^{\frac{1}{3}}}{2} \end{array} \\
E(2\pi - \delta, const + 2\pi - 2\delta) &
\end{array}$$

So the lengths of the curves connecting $E(\delta, const)$ and $E(2\pi-\delta, const+2\pi-2\delta)$ sum up to

$$\begin{aligned}
\ell_\delta \; &< \; 4\left|\cos\tfrac{\delta}{2} - \cos\tfrac{(2a\tan\frac{\delta}{2})^{\frac{1}{3}}}{2}\right| + 4\left|\cos\tfrac{2\pi-\delta}{2} - \cos\tfrac{2\pi-\left(2a\tan\frac{\delta}{2}\right)^{\frac{1}{3}}}{2}\right| \\
&\quad +4a\tan\tfrac{\delta}{2}\sin\tfrac{(2a\tan\frac{\delta}{2})^{\frac{1}{3}}}{2} + 2a\tan\tfrac{\delta}{2}\cot\tfrac{(2a\tan\frac{\delta}{2})^{\frac{1}{3}}}{2} \\
&\quad +2\tan\tfrac{\delta}{2}\sin\tfrac{\delta}{2}\sinh 2a + 2\pi\sqrt{2}e^{-a+\ln(\sin\frac{\delta}{2})} \\
&= \; 8\left|\cos\tfrac{\delta}{2} - \cos\tfrac{(2a\tan\frac{\delta}{2})^{\frac{1}{3}}}{2}\right| + 2a\tan\tfrac{\delta}{2}\left(2\sin\tfrac{(2a\tan\frac{\delta}{2})^{\frac{1}{3}}}{2} + \cot\tfrac{(2a\tan\frac{\delta}{2})^{\frac{1}{3}}}{2}\right) \\
&\quad +2\tan\tfrac{\delta}{2}\sin\tfrac{\delta}{2}\sinh 2a + 2\pi\sqrt{2}e^{-a+\ln(\sin\frac{\delta}{2})}.
\end{aligned}$$

Now we choose $a = -\frac{1}{2}\ln(\sin\frac{\delta}{2})$. With this we have

$$2a\tan\frac{\delta}{2} = -\ln(\sin\frac{\delta}{2})\tan\frac{\delta}{2} = -\frac{1}{\cos\frac{\delta}{2}}\left(\sin\frac{\delta}{2}\ln(\sin\frac{\delta}{2})\right).$$

This however converges to zero for $\delta \longrightarrow 0$ since $\lim_{x\to 0} x\ln x = 0$. Furthermore using the rule of L'Hospital we have

$$\begin{aligned}
\lim_{x\to 0} x\cot\tfrac{x^{\frac{1}{3}}}{2} &= \lim_{x\to 0}\frac{x}{\tan\frac{x^{\frac{1}{3}}}{2}} \\
&= \lim_{x\to 0}\frac{1}{\left(1+\tan^2\frac{x^{\frac{1}{3}}}{2}\right)\frac{x^{-\frac{2}{3}}}{6}} \\
&= \lim_{x\to 0}\frac{x^{\frac{2}{3}}}{6\left(1+\tan^2\frac{x^{\frac{1}{3}}}{2}\right)} \\
&= 0.
\end{aligned}$$

Finally we obtain the result wanted:

$$\begin{aligned}
\ell_\delta \; &< \; 8\underbrace{\left|\cos\frac{\delta}{2} - \cos\frac{\overbrace{(2a\tan\frac{\delta}{2})^{\frac{1}{3}}}^{\stackrel{\delta\to 0}{\longrightarrow}0}}{2}\right|}_{\stackrel{\delta\to 0}{\longrightarrow}0} + \underbrace{2a\tan\frac{\delta}{2}\left(2\sin\frac{(2a\tan\frac{\delta}{2})^{\frac{1}{3}}}{2} + \cot\frac{(2a\tan\frac{\delta}{2})^{\frac{1}{3}}}{2}\right)}_{\stackrel{\delta\to 0}{\longrightarrow}0} \\
&\quad +2\tan\tfrac{\delta}{2}\sin\tfrac{\delta}{2}\sinh 2a + 2\pi\sqrt{2}e^{-a+\ln(\sin\frac{\delta}{2})} \\
&= \underbrace{(*)}_{\stackrel{\delta\to 0}{\longrightarrow}0} +2\tan\tfrac{\delta}{2}\sin\tfrac{\delta}{2}\sinh\left(-\ln(\sin\tfrac{\delta}{2})\right) + 2\pi\sqrt{2}e^{\frac{3}{2}\ln(\sin\frac{\delta}{2})} \\
&= \underbrace{(*)}_{\stackrel{\delta\to 0}{\longrightarrow}0} + \tan\tfrac{\delta}{2}\sin\tfrac{\delta}{2}(\sin^{-1}\tfrac{\delta}{2} - \sin\tfrac{\delta}{2}) + 2\pi\sqrt{2}\sin^{\frac{3}{2}}\tfrac{\delta}{2} \\
&\stackrel{\delta\to 0}{\longrightarrow} 0.
\end{aligned}$$

Hence the Cauchy sequenzes $E(\delta, const)$ and $E(2\pi-\delta, const+2\pi-2\delta)$ converge for δ tending to zero to the same boundary point, i.e. future infinity and past infinity are identified with each other in the Cartan boundary of (M, g).

Lemma 6.10 $\overline{M}$ *is not Hausdorff and not T_1.*

Proof: Let $\overline{u}$ be the boundary point defined by the curve $E(t, const)$ for $t \longrightarrow 0$. As we have seen above for every $\varepsilon > 0$ we have a horizontal lift γ_ε^* of the light like geodesic $(t, const + t)$ such that the whole curve γ_ε^* is contained in the ε-ball around $\overline{u}$. Hence every neighbourhood of $\overline{x} := \pi(\overline{u})$ contains the whole curve $(t, const + t)$. I.e. $\overline{M}$ is not Hausdorff and not T_1.

$\square$

Actually all curves ending in the set $\{0; 2\pi\} \times \mathbb{R}$ define the same boundary point, since for example the points $E(t_n, const_1)$ and $E(t_n, const_2)$ can be connected by the horizontal curve $\gamma_n(t) := R_{A(at)} \circ E(t_n, t)$ for $t \in [const_1, const_2]$. And the length of γ_n is

$$\begin{aligned} \ell(\gamma_n) &= \sin \tfrac{t_n}{2} \underbrace{\int_{const_1}^{const_2} \sqrt{\sinh^2(-at) + \cosh^2(-at)} dt}_{=const} \\ &\overset{t_n \to 0 \text{ or } 2\pi}{\longrightarrow} 0. \end{aligned}$$

Hence all curves of the shape $(t, const)$ define the same boundary point.

6.7 Characterising the Cartan Boundary Using Embeddings

In this section we want to learn more about the Cartan boundary by embedding the studied Cartan geometry into another Cartan geometry of the same type. This section is mainly based on [Fra08].

Let $(\mathcal{G}_M, \pi_M, M; \omega_M)$ and $(\mathcal{G}_N, \pi_N, N; \omega_N)$ be two Cartan geometries of type (G, P) and

$$\sigma : (\mathcal{G}_M, \pi_M, M; \omega_M) \longrightarrow (\mathcal{G}_N, \pi_N, N; \omega_N)$$

be a geometric embedding. I.e. σ is injective and so is $d\sigma_u$ for all points $u \in \mathcal{G}_M$, furthermore $\sigma : \mathcal{G}_M \longrightarrow \sigma(\mathcal{G}_M)$ is a diffeomorphism and σ respects the right action of P ($\sigma \circ R_p = R_p \circ \sigma$), at last the pullback of the Cartan connection of N yields the Cartan connection on M ($\sigma^* \omega_N = \omega_M$).

By $s : M \longrightarrow N$ we denote the map from M to N covered by σ, $\sigma : \pi_M^{-1}(x) \longrightarrow \pi_N^{-1}(s(x))$.

If the fixed basis of $\mathfrak{g}$ defining the Riemannian metric on the bundles is the same for both Cartan geometries, σ is actually an isometry of the Riemannian manifolds, $\sigma^* \varrho_N = \varrho_M$.

$$\varrho_{M/N}\big(\omega_{M/N}^{-1}(a_i), \omega_{M/N}^{-1}(a_j)\big) := \delta_{ij} \text{ for } (a_1, \ldots, a_r) \text{ a fixed basis of } \mathfrak{g}.$$

As a subset of $\mathcal{G}_N$ the image $\sigma(\mathcal{G}_M)$ inherits the distance d_{ϱ_N}. We define another distance on $\sigma(\mathcal{G}_M)$:

$$d_{\sigma(\mathcal{G}_M)}(u, v) := \inf \left\{ \ell_{\varrho_N}(\gamma) \;\middle|\; \begin{array}{l} \gamma : [0,1] \longrightarrow \sigma(\mathcal{G}_M) \\ \gamma(0) = u, \ \gamma(1) = v \end{array} \right\}.$$

According to the definition we have $d_{\sigma(\mathcal{G}_M)} \geq d_{\varrho_N}$. So if a sequence in the image of σ is a Cauchy sequence with respect to $d_{\sigma(\mathcal{G}_M)}$ it is also one with respect to d_{ϱ_N}. Furthermore $\sigma : (\mathcal{G}_M, d_{\varrho_M}) \longrightarrow (\sigma(\mathcal{G}_M), d_{\sigma(\mathcal{G}_M)})$ is an isometry of the metric spaces.

With $\partial_{top} M$ we denote the topological boundary of $s(M) \subset N$. Analogously $\partial_{top} \mathcal{G}_M$ denotes the topological boundary of $\sigma(\mathcal{G}_M) \subset \mathcal{G}_N$.

Definition 6.5 *A point $\overline{x} \in \partial_{top}M$ is called accessible, if there is a C^1 path $\gamma : [0,1] \longrightarrow N$ with $\gamma([0,1)) \subset s(M)$ and $\gamma(1) = \overline{x}$. A point $\overline{u} \in \partial_{top}\mathcal{G}_M$ is called accessible if it is in the fiber of an accessible point of $\partial_{top}M$.*

A point $\overline{u} \in \partial_{top}\mathcal{G}_M$ being accessible is equivalent to the existence of a curve $\gamma : [0,1] \longrightarrow \mathcal{G}_N$ with $\gamma([0,1)) \subset \sigma(\mathcal{G}_M)$ and $\gamma(1) = \overline{u}$.

Lemma 6.11 *The set of accessible points is a dense subset of the topological boundary of $s(M) \subset N$ if the topological boundary $\partial_{top}M$ is nonempty.*

Proof: Given $\overline{x} \in \partial_{top}M$ we choose a point $\overline{u} \in \partial_{top}\mathcal{G}_M$ in the fibre over $\overline{x}$. Let us furthermore choose an ε-ball around $0_{\overline{u}} \in T_{\overline{u}}\mathcal{G}_N$ such that $\exp : B(0_{\overline{u}}, \varepsilon) \longrightarrow \exp\big(B(0_{\overline{u}}, \varepsilon)\big) \subset \mathcal{G}_N$ is a diffeomorphism. Since $\overline{u}$ is a boundary point of $\sigma(\mathcal{G}_M)$ in $\mathcal{G}_N$ we have a vector $X_0 \in B(0_{\overline{u}}, \varepsilon)$ with $\widetilde{X_0}\big((0,1]\big) \cap \sigma(\mathcal{G}_M) \neq \emptyset$, where $\widetilde{X_0}$ is the geodesic with speed X_0, that is $\widetilde{X_0}(t) := \exp(tX_0)$.
If there is a $t_0 \in (0,1]$ such that $\widetilde{X_0}\big((0,t_0]\big)$ is actually a subset of $\sigma(\mathcal{G}_M)$, $\overline{u}$ is already an accessible point and so is $\overline{x}$. If there is no such t_0 the open set $\widetilde{X_0}\big((0,1)\big) \cap \sigma(\mathcal{G}_M)$ is composed of connected components of the shape $\widetilde{X_0}\big((t^0_{2n+1}, t^0_{2n})\big)$ for a strictly decreasing sequence $(t^0_n) \subset (0,1)$ and every point $\widetilde{X_0}(t^0_{2n+1})$ is an accessible point of $\partial_{top}\mathcal{G}_M$. This sequence (t^0_n) might be infinite or finite. If (t^0_n) is an infinite sequence converging to zero the points $\widetilde{X_0}(t^0_{2n+1})$ converge to $\overline{u}$. Otherwise we set $\overline{u}_1 := \widetilde{X_0}(t^0_1)$ and take a vector $X_1 \in B(0_{\overline{u}}, \varepsilon^2)$ (choose in the beginning $\varepsilon < 1$) with $\widetilde{X_1}\big((0,1]\big) \cap \sigma(\mathcal{G}_M) \neq \emptyset$ and so on. Hence we obtain a sequence of accessible points of $\partial_{top}\mathcal{G}_M$ converging to $\overline{u}$. Projection results in a sequence of accessible points of $\partial_{top}M$ converging to $\overline{x}$.
Therefore, as stated the set of accessible points is a dense subset of the topological boundary of $s(M) \subset N$ given $\partial_{top}M \neq \emptyset$.

□

We now want to define the regular set of a Cartan boundary which will be useful later on.

Definition 6.6 *Let $\sigma : (\mathcal{G}_M, \pi_M, M; \omega_M) \longrightarrow (\mathcal{G}_N, \pi_N, N; \omega_N)$ be a geometric embedding. The regular set of the boundary of $\mathcal{G}_M$ associated to σ is defined as*

$$\Lambda_C := \left\{ \overline{u} \in \partial_{CB}\mathcal{G}_M \;\middle|\; \begin{array}{c} \overline{u} \textit{ is defined by a Cauchy sequence } (u_n) \subset \mathcal{G}_M \\ \textit{such that } \sigma(u_n) \textit{ converges in } \mathcal{G}_N \end{array} \right\}.$$

Of course Λ_C is P-invariant since σ is P-equivariant. Hence we can define the regular set of the Cartan boundary of M, $\lambda_C := \Lambda_C/_P$.

Proposition 6.4 *Let $\sigma : (\mathcal{G}_M, \pi_M, M; \omega_M) \longrightarrow (\mathcal{G}_N, \pi_N, N; \omega_N)$ be a strict geometric embedding of two Cartan geometries of the same type (G,P), that is to say $\partial_{top}\mathcal{G}_M$ is nonempty. Then we have:*

1. *The regular set Λ_C of $\partial_{CB}\mathcal{G}_M$ is a nonempty open subset of $\partial_{CB}\mathcal{G}_M$.*
2. *If $\overline{u} \in \Lambda_C$ is a regular point then for any sequence $(u_n) \subset \mathcal{G}_M$ tending to $\overline{u}$ the sequence $\big(\sigma(u_n)\big)$ converges in $\mathcal{G}_N$. The limit $\lim_{n\to\infty} \sigma(u_n) =: \partial\sigma(\overline{u})$ depends only on $\overline{u}$. Hence we obtain a well defined map $\partial\sigma : \Lambda_C \longrightarrow \partial_{top}\mathcal{G}_M \subset \mathcal{G}_N$.*
3. *The map $\overline{\sigma} : \Lambda_C \cup \mathcal{G}_M \longrightarrow \partial_{top}\mathcal{G}_M \cup \sigma(\mathcal{G}_M) \subset \mathcal{G}_N$ composed of σ and $\partial\sigma$ is P-equivariant and continuous. The maps $\partial\sigma$ and $\overline{\sigma}$ induce continuous maps $\partial s : \lambda_C \longrightarrow \partial_{top}M$ and $\overline{s} : \lambda_C \cup M \longrightarrow \partial_{top}M \cup s(M)$.*

   ∂s is called the boundary map of s.

4. *All accessible points of $\partial_{top}M$ are contained in $\partial s(\lambda_C)$. In the same way all accessible points of $\partial_{top}\mathcal{G}_M$ are contained in $\partial\sigma(\Lambda_C)$. In particular the images of ∂s and $\partial\sigma$ are dense in $\partial_{top}M$ resp. $\partial_{top}\mathcal{G}_M$.*

5. *The right action of P on $\Lambda_C \cup \mathcal{G}_M$ is free and proper. In particular M together with the regular points of its Cartan boundary $\lambda_C \cup M \subset \overline{M} = M \cup \partial_{CB}M$ is Hausdorff.*

The following picture illustrates this proposition.

$$\begin{tikzcd}
\partial_{CB}\mathcal{G}_M \arrow[d, phantom, "\cup"', "\text{nonempty, open subset if } \partial_{top}\mathcal{G}_M \neq \emptyset"] & & & \\
\Lambda_C \arrow[rr, "\partial\sigma"] \arrow[d, phantom, "\cap"] & & \partial_{top}\mathcal{G}_M \arrow[d, phantom, "\cap"] & \\
\overline{\mathcal{G}_M} \supset \mathcal{G}_M \dot\cup \Lambda_C \arrow[d] \arrow[dr, phantom, "\circlearrowright P \circlearrowleft"] & \supset \mathcal{G}_M \arrow[r, hook, "\sigma", "\text{geom. emb.}"'] \arrow[d] & \mathcal{G}_N \arrow[d] & \\
\underset{\text{Hausdorff}}{\overline{M} \supset M \dot\cup \lambda_C} \arrow[dd, phantom, "\cup"] & \supset M \arrow[r, hook, "s"] & N \arrow[d, phantom, "\cup"] & \\
 & & \partial_{top}M \arrow[d, phantom, "\cup"] & \begin{array}{l}\text{dense if } \partial_{top}M \neq \emptyset \\ \supset \{\text{accessible points}\}\end{array} \\
\lambda_C \arrow[rr, "\partial s"] & & \partial s(\lambda_C) &
\end{tikzcd}$$

Proof:

1. We want to prove that the regular set Λ_C is nonempty and open. Since the embedding σ is supposed to be strict, the topological boundary $\partial_{top}M$ is nonempty and according to the lemma above the set of all accessible points is dense in $\partial_{top}M$. So we have a point $\overline{u} \in \partial_{top}\mathcal{G}_M$ which is accessible. Hence we have a C^1 curve $\gamma : [0,1] \longrightarrow \mathcal{G}_N$ with $\gamma([0,1)) \subset \sigma(\mathcal{G}_M)$ and $\gamma(1) = \overline{u}$. For any sequence $(t_k) \subset [0,1)$ tending to 1 we have $d_{\sigma(\mathcal{G}_M)}\big(\gamma(t_k), \gamma(t_{k+p})\big) \leq \ell_{\varrho_N}(\gamma|_{[t_k,t_{k+p}]}) \overset{k\to\infty}{\longrightarrow} 0$. Consequently $\big(\gamma(t_k)\big)$ is a Cauchy sequence for $d_{\sigma(\mathcal{G}_M)}$. Let $u_k \in \mathcal{G}_M$ be the points with $\sigma(u_k) = \gamma(t_k)$. Then (u_k) is a Cauchy sequence in $\mathcal{G}_M$ with respect to d_{ϱ_M}. Its limit in $\partial_{CB}\mathcal{G}_M$ is denoted by $\hat{u}$. According to the construction we have $\hat{u} \in \Lambda_C$, since $\sigma(u_k) = \gamma(t_k) \overset{k\to\infty}{\longrightarrow} \overline{u} \in \mathcal{G}_N$. So Λ_C is nonempty.

   Now we will prove that the regular set Λ_C of $\partial_{CB}\mathcal{G}_M$ is open. We choose an $\varepsilon > 0$ small enough for the closure of the ε-ball around $\overline{u} \in \partial_{top}\mathcal{G}_M$ with respect to the distance d_{ϱ_N}, $\overline{B_{d_{\varrho_N}}(\overline{u}, \varepsilon)} := \{u \in \mathcal{G}_N \mid d_{\varrho_N}(\overline{u}, u) \leq \varepsilon\} \subset \mathcal{G}_N$, being complete. Furthermore let $\hat{v}$ be an element of the open subset $B_{d_{\varrho_M}}(\hat{u}, \frac{\varepsilon}{2}) \cap \partial_{CB}\mathcal{G}_M \subset \partial_{CB}\mathcal{G}_M$ of the boundary of $\mathcal{G}_M$ and choose a sequence $(v_k) \subset \mathcal{G}_M$ converging to $\hat{v}$. Then $\sigma(v_k)$ is a Cauchy sequence for $d_{\sigma(\mathcal{G}_M)}$ und therefore also for d_{ϱ_N}. For k being sufficiently big we have $\sigma(v_k) \in \overline{B_{d_{\varrho_N}}(\overline{u}, \varepsilon)}$ which is complete. Consequently $\big(\sigma(v_k)\big)$ converges in $\mathcal{G}_N$ and $\hat{v} = \lim_{k\to\infty} v_k \in \Lambda_C$ is regular. Since $\hat{v}$ was an arbitrarily chosen point in the ball $B_{d_{\varrho_M}}(\hat{u}, \frac{\varepsilon}{2}) \cap \partial_{CB}\mathcal{G}_M$ we have $B_{d_{\varrho_M}}(\hat{u}, \frac{\varepsilon}{2}) \cap \partial_{CB}\mathcal{G}_M \subset \Lambda_C \subset \partial_{CB}\mathcal{G}_M$. So the set of all regular points is also open in $\partial_{CB}\mathcal{G}_M$.

2. Let $\hat{u} \in \Lambda_C$ be a regular point, i.e. we have a Cauchy sequenze $(u_k) \subset \mathcal{G}_M$ converging to $\hat{u}$ such that the sequence $\big(\sigma(u_k)\big)$ converges in $\overline{\sigma(\mathcal{G}_M)}$. Let $(\widetilde{u}_k) \subset \mathcal{G}_M$ be another Cauchy sequence tending to $\hat{u}$. Then we have

$$\begin{array}{rcl} d_{\varrho_N}\big(\sigma(u_k), \sigma(\widetilde{u}_k)\big) & \leq & d_{\sigma(\mathcal{G}_M)}\big(\sigma(u_k), \sigma(\widetilde{u}_k)\big) \\ & = & d_{\varrho_M}(u_k, \widetilde{u}_k) \\ & \overset{k\to\infty}{\longrightarrow} & 0. \end{array}$$

So both of the sequences $\big(\sigma(u_k)\big)$ and $\big(\sigma(\widetilde{u}_k)\big)$ have the same limit in $\overline{\sigma(\mathcal{G}_M)}$, denoted by $\partial\sigma(\hat{u}) \in \partial_{top}\mathcal{G}_M$. Hence the map

$$\begin{array}{rcl} \partial\sigma : \Lambda & \longrightarrow & \partial_{top}\mathcal{G}_M \\ \hat{u} = \lim_{k\to\infty} u_k & \mapsto & \lim_{k\to\infty} \sigma(u_k) \end{array}$$

is well defined.

3. The P-equivariance of $\overline{\sigma}$ follows directly from the P-equivariance of σ and the continuity of the P-action.

$$\begin{array}{rcl} R_p \circ \overline{\sigma}(\hat{u}) & = & R_p \lim_{k\to\infty} \sigma(u_k) \\ & = & \lim_{k\to\infty} R_p \circ \sigma(u_k) \\ & = & \lim_{k\to\infty} \sigma \circ R_p(u_k) \\ & = & \partial\sigma\big(\lim_{k\to\infty} R_p(u_k)\big) \\ & = & \overline{\sigma}(R_p\hat{u}) \end{array}$$

Next we show that $\overline{\sigma}$ is also continuous. For two points $\hat{u}^1, \hat{u}^2 \in \Lambda_C \cup \mathcal{G}_M$ being represented by Cauchy sequences (u_k^1) and (u_k^2) we can write

$$\begin{array}{rcl} d_{\varrho_N}\big(\overline{\sigma}(\hat{u}^1), \overline{\sigma}(\hat{u}^2)\big) & = & \lim_{k\to\infty} d_{\varrho_N}\big(\sigma(u_k^1), \sigma(u_k^2)\big) \\ & \leq & \lim_{k\to\infty} d_{\sigma(\mathcal{G}_M)}\big(\sigma(u_k^1), \sigma(u_k^2)\big) \\ & = & \lim_{k\to\infty} d_{\varrho_M}\big(u_k^1, u_k^2\big) \\ & = & \overline{d}_{\varrho_M}(\hat{u}^1, \hat{u}^2). \end{array}$$

Hence the map $\overline{\sigma}$ is Lipschitz continuous with respect to the distances d_{ϱ_N} and $\overline{d}_{\varrho_M}$ and is therefore continuous as well. Projection and restriction results in the continuous maps $\overline{s} : \lambda_C \cup M \longrightarrow \partial_{top}M \cup s(M)$ and $\partial s : \lambda_C \longrightarrow \partial_{top}M$, the latter being called the boundary map of s.

4. Given an accessible point $\overline{u} \in \partial_{top}\mathcal{G}_M$ we want to show, that it is also contained in the image of $\partial\sigma$. According to the definition of accessible points we have a curve $\gamma : [0,1] \longrightarrow \mathcal{G}_N$ with $\gamma([0,1)) \subset \sigma(\mathcal{G}_M)$ and $\gamma(1) = \overline{u}$. That means we have a sequence $(u_k) \subset \mathcal{G}_M$ tending to a boundary point $\hat{u} \in \partial_{CB}\mathcal{G}_M$ such that the limit of the corresponding sequence $\sigma(u_k)$ is exactly the given point $\overline{u}$. Hence $\partial\sigma(\hat{u}) = \overline{u}$. So all accessible points are contained in the image of $\partial\sigma$.

   According to the lemma above the set of all accessible points is dense in $\partial_{top}\mathcal{G}_M$. Consequently the same holds for the image of $\partial\sigma$.

   The P-equivariance gives the same results for ∂s, the image $\partial s(\lambda_C)$ contains all accessible points and is dense in $\partial_{top}M$.

5. We want to show that the action of P on $\Lambda_C \cup \mathcal{G}_M$ is proper and free. These properties are directly inherited from the P-action on $\mathcal{G}_N$ with the help of $\overline{\sigma}$. Recall the definition of a group of homeomorphisms acting properly:

Definition 6.7 *Let G be a group acting on M by homeomorphisms. The action of G is called proper if for every compact subset $K \subset M$ the set $G_K := \{g \in G \mid g(K) \cap K \neq \emptyset\}$ has compact closure in $Hom(M)$. $Hom(M)$ is the group of all homeomorphisms of M endowed with the compact-open topology. The compact-open topology is the smallest topology that contains all sets $\mathcal{U}(K, U) := \{f \in C(M, M) \mid f(K) \subset U\}$ with $K \subset M$ being compact and $U \subset M$ being open.*

Remark 6.7 *If we have given a smooth manifold N, a Lie group G and a smooth, free and proper right action $\mu : N \times G \longrightarrow N$, then N is a smooth G-principal bundle.*

Let $K \subset \Lambda_C \cup \mathcal{G}_M$ be compact, $P_K := \{p \in P \mid R_p K \cap K \neq \emptyset\}$. We have to prove that the closure of P_K is compact. Since the right action of P commutes with $\overline{\sigma}$ it is $P_K = P_{\overline{\sigma}(K)}$ which has compact closure in $Hom(\mathcal{G}_N)$ because of P acting properly on $\mathcal{G}_N$. With the help of $\overline{\sigma}$ the homeomorphisms of $\Lambda_c \cup \mathcal{G}_M$ can be considered as restrictions of elements of $Hom(\mathcal{G}_N)$. Consequently P_K has also compact closure in $Hom(\Lambda_C \cup \mathcal{G}_M)$. So the action of P on $\Lambda_C \cup \mathcal{G}_M$ is proper.

To prove that the action is also free let $R_p u = u$ for some $u \in \Lambda_C \cup \mathcal{G}_M$. Hence $R_p \overline{\sigma}(u) = \overline{\sigma}(u)$. Due to P acting freely on $\mathcal{G}_N$ we have $p = e$ and therefore P also acts freely on $\Lambda_C \cup \mathcal{G}_M$.

The last point to prove is that the manifold M joined by the regular set, $M \cup \lambda_C$, is Hausdorff. Assume that $\lambda_C \cup M = \pi(\Lambda_C \cup \mathcal{G}_M)$ is not Hausdorff, that is to say we have a point $x \in \lambda_C \cup M$ with $\bigcap_{U(x)} cl(U(x)) \neq \{x\}$. Hence there ist a point $y \in \lambda_C \cup M \setminus \{x\}$ which is contained in the closure of every open neighbourhood of x. Choose $u_x \in \pi^{-1}(x)$. For every $\varepsilon > 0$ the closed ball around u_x with radius ε is compact and the projection of the open ball is an open neighbourhood of x. So we have $y \in \pi\big(cl(B_\varepsilon(u_x))\big)$. For every $k \in \mathbb{N}$ we choose a $u_k \in \pi^{-1}(y) \cap cl\big(B_{\frac{1}{k}}(u_x)\big)$. So we obtain a sequence $(u_k) \subset B_1(u_x)$ with $\pi(u_k) = y$ and this sequence converges to u_x. Furthermore we have a uniquely defined sequence $(p_k) \subset P$ with $R_{p_k} u_1 = u_k$. However this implies $R_{p_k}\big(cl(B_1(u_1))\big) \cap cl(B_1(u_x)) \neq \emptyset$ and therefore $(p_k) \subset P_{cl(B_1(u_x))}$. Since the action of P on $\Lambda_C \cup \mathcal{G}_M$ is proper, the sequence (p_k) contains a converging subsequence, i.e. there is a $p \in P$ with $R_p u_1 = \lim_{k \to \infty} R_{p_k} u_1 = \lim_{k \to \infty} u_k = u_x$. This is a contradiction to u_x and u_1 being elements of different fibres. Consequently $\lambda_C \cup M$ is Hausdorff.

□

Remark 6.8 *As we have seen in Subsection 6.2.2 the Cartan boundary of the conformally flat space $\mathbb{R}^n$ is a point. So the regular set λ_C of the Cartan boundary $\partial_{CB}\mathbb{R}^n$ contains at most one point. According to the proposition above for any strict conformal embedding $s : (\mathbb{R}^n, \langle \cdot, \cdot \rangle_n) \longrightarrow (N, g)$ the image $\overline{s}(\lambda_C)$ has to be dense in the topological boundary. Hence the topological boundary contains at most one point and is nonempty since the embedding is supposed to be strict. Therefore, (N, g) has to be conformally equivalent to the round sphere. More general knowing that the Cartan boundary $\partial_{CB} M$ consists of a finite number of points implies that the regular set λ_C and therefore also the topological boundary $\partial_{top} M$ for any strict embedding contain a finite number of points as well.*

6.7.1 The Cartan Geometry of $_\Gamma\backslash M$

In this subsection we want to describe the Cartan geometry of a manifold $_\Gamma\backslash M$, where $M \subset L$ is an open subset of a manifold L endowed with the Cartan geometry $(\mathcal{G}_L, \pi, L, \omega_L)$ and Γ is a discrete subgroup of the automorphisms of L preserving M.

Let $(\mathcal{G}_L, \pi, L; \omega_L)$ be a Cartan geometry of type (G, P). A diffeomorphism of L is called an automorphism of the Cartan geometry if it lifts to a bundle automorphism preserving the Cartan connection. Assume there is a diffeomorphism $\phi : L \longrightarrow L$ of L which has two lifts $\hat{\phi}, \widetilde{\phi} : \mathcal{G}_L \longrightarrow \mathcal{G}_L$ preserving the Cartan connection. Then we have a map $p : \mathcal{G}_L \longrightarrow P$ with $\hat{\phi}(u) = R_{p(u)} \circ \widetilde{\phi}(u)$. Hence we can write

$$\begin{aligned}
\omega(X_u) &\overset{\hat{\phi}\text{ preserves }\omega}{=} \omega\big(d\hat{\phi}(X_u)\big) \\
&= \omega\left(dR_{p(u)} \circ d\widetilde{\phi}(X_u) + \widetilde{dL_{p(u)^{-1}} \circ dp}(X_u)\right) \\
&\overset{\widetilde{\phi}\text{ preserves }\omega}{=} Ad(p(u)^{-1}) \circ \omega\left(d\widetilde{\phi}(X_u)\right) + dL_{p(u)^{-1}} \circ dp(X_u) \\
&= Ad(p(u)^{-1}) \circ \omega(X_u) + \underbrace{dL_{p(u)^{-1}} \circ dp(X_u)}_{\in\mathfrak{p}}.
\end{aligned}$$

Since the group P is supposed to act faithfully on $G/_P$ by conjugation this is only true for $p = e$ and the lifts $\hat{\phi}$ and $\widetilde{\phi}$ are identical. We denote by $Aut(L)$ the group of all automorphisms of L and the lift of an automorphism $\phi : L \longrightarrow L$ is denoted with the same symbol $\phi : \mathcal{G}_L \longrightarrow \mathcal{G}_L$.

In order to see that $Aut(L)$ acts freely on $\mathcal{G}_L$ we use the following lemma.

Lemma 6.12 *Let $(\mathcal{G}_M, \pi_M, M; \omega_M)$ and $(\mathcal{G}_N, \pi_N, N; \omega_N)$ be two Cartan geometries, $\mathcal{G}_M$ connected. Let furthermore $f_1, f_2 : \mathcal{G}_M \longrightarrow \mathcal{G}_N$ be two diffeomorphisms respecting the Cartan connections, that is $f_{1/2}^*\omega_N = \omega_M$, and satisfying $f_1(u_0) = f_2(u_0)$ and $(df_1)_{u_0} = (df_2)_{u_0}$ for a point $u_0 \in \mathcal{G}_M$. Then we have $f_1 \equiv f_2$.*

Proof: Let A be the set of all points of $\mathcal{G}_M$ where the two maps and their differentials are identical,

$$A := \{u \in \mathcal{G}_M \mid f_1(u) = f_2(u), (df_1)_u = (df_2)_u\}.$$

We have $u_0 \in A$ and A is closed, since $f_{1/2}$ and $df_{1/2}$ are continuous. So we have to prove that A is also open. Then A is equal to $\mathcal{G}_M$ which proves the statement. So let u be a point in A. For a vector $V \in T_u\mathcal{G}_M$ we have the integral curve γ_V of the ω-constant vector field $\omega_M^{-1} \circ (\omega_M)_u(V)$ through the point u. For the curves $\gamma_{1/2} := f_{1/2} \circ \gamma_V$ we have $\gamma_1(0) = f_1 \circ \gamma_V(0) = f_1(u) = f_2(u) = \gamma_2(0)$ and

$$\begin{aligned}
\dot{\gamma}_{1/2}(t) &= df_{1/2} \circ \dot{\gamma}_V(t) \\
&= df_{1/2} \circ (\omega_M)^{-1}_{\gamma_V(t)} \circ (\omega_M)_u(V) \\
&= (\omega_N)^{-1} \circ (\omega_M)_u(V).
\end{aligned}$$

I.e. $\gamma_{1/2}$ are integral curves of the same vector field with the same starting point and they are therefore identical, hence $f_1 = f_2$ along the curve γ_V. Since such integral curves are defined on an open neighbourhood of u we have $f_1 = f_2$ on an open neighbourhood of u inducing the equality of their derivatives there. So A is also open and therefore equal to M and the two diffeomorphisms are the same.

$\square$

With this lemma it is easy to see that $Aut(L)$ acts freely on $\mathcal{G}_L$. Let $\phi \in Aut(L)$ be an automorphism with $\phi(u_0) = u_0$. Since $d\phi$ preserves the Cartan connection and therefore also the parallelisation $(A_i)_u := \omega_u^{-1}(a_i)$ defined by the Cartan connection we have at the fixed point u_0

$$d\phi_{u_0}\big((A_i)_{u_0}\big) = (A_i)_{\phi(u_0)} = (A_i)_{u_0}.$$

I.e. $d\phi_{u_0} = id$. So according to the lemma above ϕ is actually the identity and $Aut(L)$ acts freely on $\mathcal{G}_L$.
Now let $M \subset L$ be an open subset. The Cartan bundle of M is given by $\pi^{-1}(M)$ and its Cartan connection is the restriction of the Cartan connection of L. As we have seen, fixing a basis $(a_1, \ldots, a_r)$ of $\mathfrak{g}$ leads to Riemannian metrics ϱ_L on $\mathcal{G}_L$ and ϱ_M on $\mathcal{G}_M$, where ϱ_M is just the restiction of the Riemannian metric on $\mathcal{G}_L$.
Let Γ be a discrete subgroup of the automorphisms of L preserving M. On the quotient manifold ${}_\Gamma\backslash\mathcal{G}_L$ the subgroup P acts from the right. However ${}_\Gamma\backslash\mathcal{G}_L$ is not necessarily a P-principal bundle.
We will take a closer look under what conditions we actually obtain a P-principal bundle.

Lemma 6.13 *Let $\Gamma \subset Aut(L)$ be a discrete subgroup of automorphisms acting properly on L. Then Γ also acts properly on $\mathcal{G}_L$.*

Proof: Let $\mathcal{K} \subset \mathcal{G}_L$ be compact. We have to prove, that $\Gamma_\mathcal{K}$ has compact closure.
$K := \pi(\mathcal{K}) \subset L$ is compact. For every automorphism $\phi \in \Gamma_\mathcal{K} = \{\phi \in \Gamma \mid \phi(\mathcal{K}) \cap \mathcal{K} \neq \emptyset\}$ there is a $u_x \in \mathcal{K}$ such that $\phi(u_x) \in \mathcal{K}$. However this implies $\phi(x) \in K$. So $\phi(K) \cap K \neq \emptyset$ and ϕ is also an element of Γ_K. Hence $\Gamma_\mathcal{K} \subset \Gamma_K$. We know that Γ_K has compact closure since Γ is acting properly on L. So the closure of $\Gamma_\mathcal{K}$ is compact as well. Consequently Γ is acting properly on $\mathcal{G}_L$.

□

Lemma 6.14 *Let $\Gamma \subset Aut(L)$ be a discrete subgroup of automorphisms. P acts properly on ${}_\Gamma\backslash\mathcal{G}_L$ if Γ acts properly on L.*

Proof: Let Γ act properly on L and let $\mathcal{K}$ be a compact subset of ${}_\Gamma\backslash\mathcal{G}_L$. We have a compact subset $K \subset \mathcal{G}_L$ such that ${}_\Gamma\backslash K = \mathcal{K}$. Now we can write

$$P_\mathcal{K} = \big\{p \in P \bigm| \exists\gamma \in \Gamma : R_p K \cap \gamma \cdot K \neq \emptyset\big\}.$$

However since Γ is discrete and acts properly on L the subsets K and $\gamma \cdot K$ will have a nonempty intersection solely for a finite number of elements $\gamma \in \Gamma$. More precisely for compact $K \subset \mathcal{G}_L$ the set $P_K^\Gamma = \{\gamma \in \Gamma \mid \gamma \cdot K \cap K \neq \emptyset\}$ has compact closure and is therefore finite since Γ is discrete.
Thus it is

$$\begin{aligned} P_\mathcal{K} &= \textstyle\bigcup_{\gamma \in P_K^\Gamma} \big\{p \in P \bigm| R_p K \cap \gamma \cdot K \neq \emptyset\big\} \\ &\subset P_{\bigcup_{\gamma \in P_K^\Gamma} \gamma \cdot K}. \end{aligned}$$

With P_K^Γ being finite the subset $\bigcup_{\gamma \in P_K^\Gamma} \gamma \cdot K \subset \mathcal{G}_L$ is compact. And due to P acting properly on $\mathcal{G}_L$ the closure of $P_{\bigcup_{\gamma \in P_K^\Gamma} \gamma \cdot K}$ is compact. Consequently also $P_\mathcal{K}$ has a compact closure and P acts properly on ${}_\Gamma\backslash\mathcal{G}_L$.

□

Lemma 6.15 *Let $\Gamma \subset Aut(L)$ be a discrete subgroup of automorphisms. P acts freely on the fibres of ${}_\Gamma\backslash\mathcal{G}_L$ if Γ acts freely on L.*

Proof: We will actually prove the contraposition of this lemma.
Assume that P is not acting freely on the fibres of ${}_\Gamma\backslash\mathcal{G}_L$. I.e. we have a point $[u_x] \in_\Gamma \backslash\mathcal{G}_L$ and a nontrivial $p \in P$ such that $R_p[u_x] = [R_p u_x] = [u_x]$. So we have a nontrivial $\phi \in \Gamma$ with $\phi(R_p u_x) = u_x$. However by projection this implies $\phi(x) = x$ and hence Γ does not act freely on L.

$\square$

From the lemmata above we know that for Γ being a discrete subgroup of the automorphisms of L acting freely and properly on L we actually obtain a P-principal bundle ${}_\Gamma\backslash\mathcal{G}_L$. In the same way if M is an open subset of L and Γ a discret subgroup of $Aut(L)$ preserving M and acting freely and properly on it, ${}_\Gamma\backslash\mathcal{G}_M$ is a P-principal bundle over ${}_\Gamma\backslash M$.
We now want to describe the Cartan connection of $({}_\Gamma\backslash\mathcal{G}_M, \pi, {}_\Gamma\backslash M; P)$ in terms of the Cartan connection of $(\mathcal{G}_L, \pi_L, L; P)$.
With $\omega_{\Gamma\backslash L}$ we denote the 1-form induced by ω_L,

$$(\omega_{\Gamma\backslash L})_{[u]}([X]) := (\omega_L)_u(X) \text{ with } [X] = [X_u] = \{d\phi_u(X_u) \mid \phi \in \Gamma\}.$$

This is well defined since the automorphisms in Γ preserve the Cartan connection, that is to say for $\phi \in \Gamma$ we have $\phi^*\omega_L = \omega_L$. Furthermore we have

- $\omega_{\Gamma\backslash L}$ is P-equivariant,

$$R_p^*\omega_{\Gamma\backslash L}([X]) = R_p^*\omega_L(X) = Ad(p^{-1}) \circ \omega_L(x) = Ad(p^{-1}) \circ \omega_{\Gamma\backslash L}([X]).$$

- $\omega_{\Gamma\backslash L}$ reproduces the generators of the fundamental vector fields,

$$\omega_{\Gamma\backslash L}([\widetilde{X}]) = \omega_L(\widetilde{X}) = X.$$

- For every point $[u] \in_\Gamma \backslash\mathcal{G}_L$ we have an isomorphism $(\omega_{\Gamma\backslash L})_{[u]} : T_{[u]}\,({}_\Gamma\backslash\mathcal{G}_L) \longrightarrow \mathfrak{g}$.

Definition 6.8 *For a Cartan geometry $(\mathcal{G}_L, \pi_L, L; \omega_L)$, an open subset $M \subset L$ and a discret subgroup $\Gamma \subset Aut(L)$ of automorphisms preserving M and acting freely and properly on it $({}_\Gamma\backslash\mathcal{G}_L, \pi, {}_\Gamma\backslash L; \omega_{\Gamma\backslash L})$ is called a weak Cartan geometry. If Γ acutally acts freely and properly on L itself $({}_\Gamma\backslash\mathcal{G}_L, \pi, {}_\Gamma\backslash L; \omega_{\Gamma\backslash L})$ is a Cartan geometry.*

With $\omega_{\Gamma\backslash M}$ we denote the restiction of $\omega_{\Gamma\backslash L}$ to the bundle ${}_\Gamma\backslash\mathcal{G}_M$. So we have a Cartan connection $\omega_{\Gamma\backslash M}$ on the principal bundle $({}_\Gamma\backslash\mathcal{G}_M, \pi, {}_\Gamma\backslash M; P)$.

Definition 6.9 *Let $(\mathcal{G}_L, \pi_L, L; \omega_L)$ be a Cartan geometry and $M \subset L$ an open subset. For a discrete subgroup $\Gamma \subset Aut(L)$ preserving M and acting freely and properly on it $({}_\Gamma\backslash\mathcal{G}_M, \pi, {}_\Gamma\backslash M; \omega_{\Gamma\backslash M})$ is a Cartan geometry and is called the canonical Cartan geometry on ${}_\Gamma\backslash M$, induced by the Cartan geometry of L.*

On $\mathcal{G}_L$ we have the Riemannian metric ϱ_L defined by the Cartan connection ω_L. This metric induces the Riemannian metric $\varrho_{\Gamma\backslash L}$ on ${}_\Gamma\backslash\mathcal{G}_L$. Restriction of ϱ_L to $\mathcal{G}_M$ gives the metrik $\varrho_M = \varrho_L|_{\mathcal{G}_M}$. And finaly ϱ_M induces $\varrho_{\Gamma\backslash M}$ on ${}_\Gamma\backslash\mathcal{G}_M$. This metric is identical to the Riemannian metric defined by the Cartan connection $\omega_{\Gamma\backslash M}$ of the Cartan geometry $({}_\Gamma\backslash\mathcal{G}_M, \pi, {}_\Gamma\backslash M; \omega_{\Gamma\backslash M})$.
All those Riemannian metrics define distances, $d_{\Gamma\backslash L}$ on ${}_\Gamma\backslash\mathcal{G}_L$ and $d_{\Gamma\backslash M}$ on ${}_\Gamma\backslash M$. For two points $[u], [v] \in_\Gamma \backslash\mathcal{G}_M$ we have

$$\begin{aligned} d_{\Gamma\backslash M}([u],[v]) &= \inf\{\ell(\gamma) \mid \gamma : [0,1] \longrightarrow {}_\Gamma\backslash\mathcal{G}_M, \gamma(0) = [u], \gamma(1) = [v]\} \\ &\geq \inf\{\ell(\gamma) \mid \gamma : [0,1] \longrightarrow {}_\Gamma\backslash\mathcal{G}_L, \gamma(0) = [u], \gamma(1) = [v]\} \\ &= d_{\Gamma\backslash L}([u],[v]). \end{aligned}$$

In order to determine the Cartan boundary of ${}_\Gamma\backslash M$ we need to find all Cauchy sequences of ${}_\Gamma\backslash\mathcal{G}_M$ with respect to the distance $d_{\Gamma\backslash M}$. A sequence $([u_n]) \subset_\Gamma \backslash\mathcal{G}_M$ is a Cauchy sequence with respect to $d_{\Gamma\backslash M}$ if and only if there is a sequence of automorphisms $(\phi_n) \subset \Gamma$ such that $(\phi_n(u_n)) \subset \mathcal{G}_M$ is a Cauchy sequence with respect to d_M. This implies that $(\phi_n(u_n))$ is a Cauchy sequence with respect to d_L and $([u_n])$ is one with respect to $d_{\Gamma\backslash L}$. If we also know that $\mathcal{G}_M \subset \mathcal{G}_L$ is dense, we get $\overline{{}_\Gamma\backslash\mathcal{G}_M} = \overline{{}_\Gamma\backslash\mathcal{G}_L}$. And for $\mathcal{G}_M$ being dense in the complete space $\mathcal{G}_L$ we have $\overline{{}_\Gamma\backslash\mathcal{G}_M} = {}_\Gamma\backslash\mathcal{G}_L$. This we will use for the example in the next section.

Remark 6.9 *For $\mathcal{G}_M \subset \mathcal{G}_L$ not being dense we do not have such strong results. Still we know that if $M \subset L$ is a normal domain there is a map $\varphi : \partial_{top}\left({}_\Gamma\backslash\mathcal{G}_M\right) \longrightarrow \partial_{CB}\left({}_\Gamma\backslash\mathcal{G}_M\right)$, mapping the topological boundary of ${}_\Gamma\backslash\mathcal{G}_M \subset_\Gamma \backslash\mathcal{G}_L$ homeomorphically onto an open subset of the Cartan boundary of ${}_\Gamma\backslash\mathcal{G}_M$ (see [Fra08] Sections 3.3 and 3.4).*
Here a normal domain is defined as $M \subsetneq L$ being an open subset such that for any point of the topological boundary, $y \in \partial_{top}M$, there is a countable family $(U_i)_{i\in\mathbb{N}}$ of connected relatively compact neighbourhoods with

- *$\overline{U_{i+1}} \subsetneq U_i$ for all $i \in \mathbb{N}$ and $\bigcap_{i\in\mathbb{N}} U_i = \{y\}$,*
- *$U_i \cap M$ is connected for all $i \in \mathbb{N}$ and*
- *for every smooth Riemannian metric ϱ on U_0 the metrics $d_\varrho^{U_i}$ and $d_\varrho^{U_i\cap M}$ are bi-Lipschitz on $U_i \cap M$ for every $i \in \mathbb{N}$.*

6.7.2 The Cartan Boundary of the Conformal Space ${}_\Gamma\backslash\mathbb{R}^n$

In this section we want to describe the Cartan boundary of a conformal manifold defined by a flat, connected and complete Riemannian manifold. Recall that any flat, connected, complete Riemannian manifold (M^n, g) is isometric to ${}_\Gamma\backslash\mathbb{R}^n$ for some discrete subgroup $\Gamma \subset Iso(\mathbb{R}^n)$ acting freely and properly on $\mathbb{R}^n$. In fact the boundary of ${}_\Gamma\backslash\mathbb{R}^n$ will contain only one point for Γ acting freely on $\mathbb{R}^n$, as we will see later. However we do not want to restrict to the case of Γ acting freely.
Let $\Gamma \subset Iso(\mathbb{R}^n)$ be a nontrivial and discrete subgroup of the Euklidian motions of $\mathbb{R}^n$ endowed with the standard Euklidian metric $\langle\cdot,\cdot\rangle$. Such a group is always acting properly on $\mathbb{R}^n$, since $Iso(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$. Hence for a compact and therefore bounded subset $K \subset \mathbb{R}^n$ and some $\varphi \in \Gamma_K \subset Iso(\mathbb{R}^n)$ we know that $\varphi \in O(n) \ltimes [a,b]^n$ has to hold for some numbers $a, b \in \mathbb{R}$ depending only on K. However this is a compact subset of $Iso(\mathbb{R}^n)$ and therefore Γ_K has a compact closure and Γ is acting properly on $\mathbb{R}^n$.
To deal with the case of Γ not acting freely on the whole of $\mathbb{R}^n$ we define the set of critical points,

$$Crit = \{x \in \mathbb{R}^n \mid \exists\varphi \in \Gamma, \varphi \neq id \text{ with } \varphi(x) = x\}.$$

Lemma 6.16 *$\mathbb{R}^n \setminus Crit$ is an open and dense subset of $\mathbb{R}^n$.*

Proof: Assume that $\mathbb{R}^n \setminus Crit$ is not open, i.e. we have a point $x \in \mathbb{R}^n \setminus Crit$ and a sequence of critical points $(x_k) \subset Crit$ converging to x. Furthermore we have a sequence of automorphisms $\varphi_k \in \Gamma \setminus \{id\}$ with $\varphi_k(x_k) = x_k$. Using coordinates those automorphism can be written as $\varphi_k(*) = A_k \cdot * + w_k$ with $A_k \in O(n)$ and $w_k \in \mathbb{R}^n$. Since the orthogonal group $O(n)$ is compact the sequence (A_k) contains a converging subsequence. Without loss of generality we assume that the whole sequence converges. With (x_k) and (A_k) converging and $A_k \cdot x_k + w_k = x_k$ we obtain that (w_k) converges as well and so does (φ_k). Since

Γ is discrete this implies that there is a k_0 such that $(\varphi_k)_{k\geq k_0}$ is constant and therefore $\varphi_{k_0}(x) = x$. This is a contradiction to x being not critical. Hence $\mathbb{R}^n \setminus Crit$ is open.
Let us now assume, that $\mathbb{R}^n \setminus Crit$ is not dense in $\mathbb{R}^n$, that is there is a ball with radius $\varepsilon > 0$ containing only critical points, $B_\varepsilon(x) \subset Crit$. So the following points $x_k^{e_j} := x + \frac{\varepsilon}{k} e_j$ and $x_k^{-e_j} := x - \frac{\varepsilon}{k} e_j$ are critical and are preserved by automorphisms $\varphi_k^{\pm e_j} \in \Gamma \setminus \{id\}$, $\varphi_k^{\pm e_j}(x_k^{\pm e_j}) = x_k^{\pm e_j}$. As above we have a k_0 such that $\varphi_k^{\pm e_j} = \varphi_{k_0}^{\pm e_j}$ for all $k \geq k_0$. The same arguments applied to the sequence $\varphi_{k_0+1}^{e_1}, \varphi_{k_0+2}^{e_2}, \dots, \varphi_{k_0+n}^{e_n}, \varphi_{k_0+n+1}^{-e_1}, \dots, \varphi_{k_0+2n}^{-e_n}, \varphi_{k_0+2n+1}^{e_1}, \dots$ imply that this is actually a constant sequence. Hence $\varphi := \varphi_{k_0+1}^{e_1} \in \Gamma \setminus \{id\}$ leaves the points $x_{k_0+1}^{e_1}, x_{k_0+2}^{e_2}, \dots, x_{k_0+n}^{e_n}, x_{k_0+n+1}^{-e_1}$ invariant. Sinces φ is an isometry every point of the convex hull is invariant under the action of φ. However this is only true for $\varphi = id$. This is a contradiction to $\varphi \in \Gamma \setminus \{id\}$ and therefore $\mathbb{R}^n \setminus Crit$ has to be dense in $\mathbb{R}^n$.

□

So on $M := \mathbb{R}^n \setminus Crit$ the discrete and nontrivial subgroup $\Gamma \subset Iso(\mathbb{R}^n)$ is acting properly and freely. With the help of the stereographic projection we can view M as an open and dense subset of the sphere S^n and the automorphisms of $\mathbb{R}^n$ can be prolonged to automorphisms of the sphere preserving the north pole. Hence from the section above we know that the Cartan geometry of the conformal space ${}_\Gamma\backslash M$ is given in terms of the Cartan geometry of the sphere. With $\mathcal{G}_M$ being the restriction of $\mathcal{G}_{S^n}$ to M and ω_M being the restriction of ω_{S^n} to the bundle $(\mathcal{G}_M, \pi, M)$ the Cartan geometry of ${}_\Gamma\backslash M$ is given as

$$\left({}_\Gamma\backslash \mathcal{G}_M, \pi, {}_\Gamma\backslash M; (\pi^\Gamma)_* \omega_M \right).$$

Since $\mathcal{G}_{S^n} = SO(1, n+1)$ is complete we obtain for the Cauchy completion of ${}_\Gamma\backslash \mathcal{G}_M$

$$\overline{\left({}_\Gamma\backslash \mathcal{G}_M \right)} = {}_\Gamma\backslash \mathcal{G}_{S^n}.$$

So we have to take a closer look at the structure of ${}_\Gamma\backslash \mathcal{G}_{S^n}$. I.e. we have to determine the lifts of the automorphisms $\varphi \in \Gamma$. In order to simplify the calculations we will determine the lifts over $\mathbb{R}^n$ using the properties of the Cartan geometry of the conformal space $\mathbb{R}^n$ which can be found in [Feh05] and then transfer the informations obtained with the help of the stereographic projection onto the bundle over the sphere.
Let $\mathcal{G}$ be the Cartan bundle over $\mathbb{R}^n$ with the structure group P. Using the standard metric of $\mathbb{R}^n$ we get the following sections.

$$\begin{array}{ccccc}
 & \mathcal{G} & & \ni ker\left(A^{\langle\cdot,\cdot\rangle}_{(e_i)_x}\right) \\
\pi^1 & \downarrow\uparrow & \sigma^{\langle\cdot,\cdot\rangle} & \\
 & CO(n) & & \ni (e_1, \dots, e_n)_x \\
\pi_1 & \downarrow\uparrow & \sigma_0 & \\
 & \mathbb{R}^n & & \ni x
\end{array}$$

σ_0 maps any point of the Euklidian space onto the standard frame. $\sigma^{\langle\cdot,\cdot\rangle}$ selects for any conformal frame the horizontal subspace defined by the Levi-Civita connection $A^{\langle\cdot,\cdot\rangle}$ of the standard metric. The composition of both is the global section $\sigma := \sigma^{\langle\cdot,\cdot\rangle} \circ \sigma_0 : \mathbb{R}^n \longrightarrow \mathcal{G}$.
We need to lift the automorphism $\varphi \in \Gamma$ to automorphism $\phi : \mathcal{G} \longrightarrow \mathcal{G}$, such that the Cartan connection is invariant under pullback with ϕ, that is to say $\phi^*\omega = \omega$.
Since φ is an automorphism of the euklidian space we can write with respect to the frame $(e_1, \dots, e_n)$

$$\varphi(x) = Ax + w, \text{ for } A \in O(n) \text{ and } w \in \mathbb{R}^n.$$

In the conformal case the Cartan connection splits into three parts $\omega = \omega_{-1} \oplus \omega_0 \oplus \omega_1$ and we have

- $\omega_{-1} = (\pi^1)^*\theta_{\mathcal{CO}(n)}$, where $\theta_{\mathcal{CO}(n)}$ denotes the displacementform.
- $\sigma_0^*(\sigma^{\langle\cdot,\cdot\rangle})^*\omega_0 = \sigma_0^* A^{\langle\cdot,\cdot\rangle} = 0$ since the global frame $(e_1, \ldots, e_n)$ is parallel with respect to the Levi Civita connection of $\langle\cdot,\cdot\rangle$.
- $(\sigma^{\langle\cdot,\cdot\rangle})^*\omega_1 = 0$ since it can be written in terms of the Schouten tensor which vanishes in the flat case.

With those informations we can write for a curve γ in $\mathbb{R}^n$

$$\begin{aligned}\sigma^*\omega\big(\dot\gamma\big) &= \sigma^*(\pi^1)^*\theta_{\mathcal{CO}(n)}(\dot\gamma)\\ &= \theta_{\mathcal{CO}(n)} \circ d\sigma_0(\dot\gamma)\\ &= [\sigma_0(\gamma)]^{-1}(\dot\gamma).\end{aligned}$$

Since $\phi^*\omega \overset{!}{=} \omega$ has to hold, the lift of φ is unique and necessarily $\phi\circ\sigma = R_A\circ\sigma\circ\varphi$. This can be seen in the following calculations.

$$\begin{aligned}\big(\phi^*\omega\big)_{\sigma(\gamma)}\big(d\sigma_\gamma(\dot\gamma)\big) &\overset{!}{=} \omega_{\sigma(\gamma)}\big(d\sigma_\gamma(\dot\gamma)\big)\\ &= [\sigma_0(\gamma)]^{-1}(\dot\gamma)\\ &= A^{-1}\cdot[\sigma_0\circ\varphi(\gamma)]^{-1}\big(d\varphi(\dot\gamma)\big)\\ &= A^{-1}\cdot\theta_{\mathcal{CO}(n)}\big(d\sigma_0\circ d\varphi(\dot\gamma)\big)\\ &= A^{-1}\cdot\omega_{\sigma\circ\varphi\circ\gamma}\big(d\sigma\circ d\varphi(\dot\gamma)\big)\\ &= Ad(A^{-1})\circ\omega_{\sigma\circ\varphi\circ\gamma}\big(d\sigma\circ d\varphi(\dot\gamma)\big)\\ &= (R_A^*\omega)_{\sigma\circ\varphi\circ\gamma}\big(d\sigma\circ d\varphi(\dot\gamma)\big)\\ &= \omega_{R_A\circ\sigma\circ\varphi\circ\gamma}\big(dR_A\circ d\sigma\circ d\varphi(\dot\gamma)\big)\\ &= \omega_{\phi\circ\sigma\circ\gamma}\big(d\phi_{\sigma\circ\gamma}\circ d\sigma_\gamma(\dot\gamma)\big)\end{aligned}$$

Since ϕ commutes with the right action of P we have for any $u_x \in \pi^{-1}(x)$ with $u_x = R_p\circ\sigma(x)$

$$\phi(u_x) = \phi\circ R_p\circ\sigma(x) = R_p\circ\phi\circ\sigma(x) = R_p\circ R_A\circ\sigma\circ\varphi(x).$$

We can view the elements of $O(n)$ as elements of $P = O(1, n+1)|_{\mathbb{R}f_-}$ by

$$\begin{aligned}O(n) &\longrightarrow P\\ A &\mapsto \begin{pmatrix}1&0&0\\0&A&0\\0&0&1\end{pmatrix}.\end{aligned}$$

For x being a critical point we define

$$\Gamma_x := \{A\in O(n)\subset P \mid \exists\varphi\in\Gamma : \varphi(x) = x \text{ and } \varphi(*) = A*+w\}.$$

So the fibre in ${}_\Gamma\backslash\mathbb{R}^n$ over x is ${}_{\Gamma_x}\backslash P$. Since the automorphisms of $\mathbb{R}^n$ are pushed foreward by the stereographic projection and prolonged to automorphisms of the sphere fixing the north pole, the fibre of ${}_\Gamma\backslash\mathcal{G}_{S^n}$ over the north pole is ${}_{\Gamma_\infty}\backslash P$ with

$$\Gamma_\infty := \{A\in P \mid \exists\varphi\in\Gamma : \varphi(*) = A*+w\}.$$

We conclude

$$\begin{aligned}\partial_{CB}\left({}_\Gamma\backslash\mathcal{G}_M\right) &= \dot{\bigcup}_{x\in Crit}\, {}_{\Gamma_x}\backslash P \,\dot\cup\, {}_{\Gamma_\infty}\backslash P \text{ and}\\ \partial_{CB}\left({}_\Gamma\backslash M\right) &= Crit\,\dot\cup\,\{point\}.\end{aligned}$$

Especially if there are no critical points, i.e. Γ is acting freely on the conformal space $\mathbb{R}^n$, we have $\partial_{CB}\left({}_{\Gamma}\backslash\mathbb{R}^n\right) = \{\text{point}\}$.
We now want to prove that the conformal space $\overline{{}_{\Gamma}\backslash\mathbb{R}^n}$ will not be Hausdorff if the subgroup Γ is not compact. We denote with $\overline{x}$ the boundary point of $\mathbb{R}^n$. Let $U(\overline{x}) \subset \overline{\mathbb{R}^n}$ be a neighbourhood of $\overline{x}$. Thus $\mathbb{R}^n \setminus U(\overline{x})$ is compact and we have a $\varphi \in \Gamma$ with $\varphi(0) \notin \mathbb{R}^n \setminus U(\overline{x})$. Consequently in the conformal space $\overline{{}_{\Gamma}\backslash\mathbb{R}^n}$ the class of the origin $[0]$ will be an element of any neighbourhood of the boundary point $[\overline{x}]$. And so $\overline{{}_{\Gamma}\backslash\mathbb{R}^n}$ will not be Hausdorff if the subgroup Γ is not compact.
For example the conformal cylinder ${}_{span\{e_1\}}\backslash\mathbb{R}^n$ has one boudary point, with fibre isomorphic to the whole group $P = CO(n) \ltimes \mathbb{R}^n$, and $\overline{{}_{span\{e_1\}}\backslash\mathbb{R}^n}$ is not Hausdorff.

6.7.3 Some Remarks on the Conformal Boundary

Another boundary physicists often use is the conformal boundary. Since this boundary is defined with the help of an embedding we now want to give a short comment on this theme. More detailed informations can be found in [Falk07] for example.

Definition 6.10 *Let $\varphi : (M, g) \longrightarrow (X, c)$ be a conformal embedding of a semi-Riemannian manifold without a boundary into a conformal manifold (X, c). Further let $\varrho : X \longrightarrow \mathbb{R}$ be a smooth map with*

- *$\varrho(x) > 0$ for all $x \in \varphi(M)$,*
- *$(\varphi_* g)_x = \frac{1}{\varrho^2}\overline{g}_x$ for a metric $\overline{g} \in c$ and all points $x \in \varphi(M) \subset X$,*
- *$\partial\big(\varphi(M)\big) = \varrho^{-1}(0)$ and*
- *$(d\varrho)_x \neq 0$ for all $x \in \partial(\varphi(M))$.*

Then $\partial(\varphi(M))$ is called conformal boundary of the semi-Riemannian manifold (M, g) with respect to (X, c) and is endowed with the conformal structure induced by $\overline{g}$.

Obviously this boundary definition is not intrinsic but depends on the embedding chosen. And in [Chr02] an example is given of a Lorentzian manifold with two conformal boundaries which are not equivalent.
However applying the results from [Fra08] to the context of conformal boundaries of conformal manifolds of dimension $n \geq 3$ we find that the image of the regular set $\partial\varphi(\lambda_C)$ under the boundary map has to be a dense subset of the conformal boundary. And with the regular set being defined intrisically as a subset of the Cartan boundary, we know that two conformal boundaries of a manifold cannot "differ to much" as they coincide on a dense subset.

Chapter 7

The Cartan Boundaries of CR Manifolds and Fefferman Spaces

In this chapter we will apply the construction of the Cartan boundary to CR manifolds and the corresponding Fefferman spaces. We will consider both constructions studied earlier.

7.1 The Construction of [BL04]

Recall that the Fefferman space was defined with the help of the canonical line bundle $\mathcal{F} := \big(\mathcal{K} \setminus \{0\}\big)\big/_{\mathbb{R}^+}$ such that $\mathcal{F} \longrightarrow M$ is a S^1-principal bundle. It is equipped with the conformal class given by the Lorentzian metric

$$h_\theta := \pi^* L_\theta - i\frac{4}{n+2}\pi^*\theta \odot A^\theta.$$

Here θ is the pseudo-hermitian form of the strictly pseudo-convex CR manifold and the connection A^θ is defined as $A^\theta := A^W - \frac{i}{2(n+1)} R^W \cdot \theta$ where R^W is the scalar curvature of the Tanaka Webster connection on M and A^W is a S^1-principal bundle connection defined by the Tanaka Webster connection.
With the help of the Cartan bundle $\underline{\mathcal{G}}_M$ of the CR manifold M we can write

$$\mathcal{F} \simeq \underline{\mathcal{G}}\big/_{(\widetilde{P}^+ \cap G)\cdot Z(G)}$$

as we have seen in Section 5.7.

7.1.1 The S^1-Action

Let us take a closer look at the Fefferman space and the S^1-action. As we have seen in Section 2.2 the fundamental vector field generated by i is a light like Killing vector field with respect to the metric h_θ. Hence S^1 acts by isometries with respect to h_θ, $f_\alpha : (\mathcal{F}, h_\theta) \longrightarrow (\mathcal{F}, h_\theta)$. Consequently we can lift the action of S^1 to the bundle of all conformal frames of $\mathcal{F}$ via

$$\begin{array}{rcl} F_\alpha : \mathcal{CO}(\mathcal{F}) & \longrightarrow & \mathcal{CO}(\mathcal{F}) \\ s_\varphi & \mapsto & (df_\alpha)_\varphi(s_\varphi) \end{array}$$

for $\alpha \in S^1$ and s_φ being a conformal frame of $T_\varphi\mathcal{F}$. We have $\pi \circ F_\alpha = f_\alpha \circ \pi$ and the action is free, since S^1 acts by isomorphisms with respect to the metric h_θ.
The Cartan bundle of the conformal space $\mathcal{F}$ is a prolongation of the conformal repere bundle with structure group $\widetilde{P}_1 \simeq (\mathbb{R}^{2n+1})^*$ (see [Feh05]) and we have the following diagramm.

$$\begin{tikzcd}
S^1 \overset{?}{\circlearrowright} \mathcal{G}_{\mathcal{F}} \circlearrowleft CO(2n+1) \ltimes (\mathbb{R}^{2n+1})^* \arrow[d, shift left, "\pi^1"] \\
S^1 \circlearrowright \mathcal{CO}(\mathcal{F}) \circlearrowleft CO(2n+1) \arrow[u, shift left, "\sigma"] \arrow[d, "\pi_1"] \\
S^1 \circlearrowright \mathcal{F} \arrow[r] & M
\end{tikzcd}$$

Now by fixing a Weyl structure $\sigma : \mathcal{CO}(\mathcal{F}) \longrightarrow \mathcal{G}_{\mathcal{F}}$, that is to say a $CO(1, 2n+1)$-equivariant global section of the projection π^1 (take for example $\sigma^\theta : \mathcal{CO}(\mathcal{F}) \ni s_\varphi \mapsto ker\big(A^{h_\theta}_{s_\varphi}\big) \in \mathcal{G}_{\mathcal{F}}$), we can define a S^1-action $F^\sigma_{S^1}$ on the Cartan bundle of the Fefferman space. For any element $u \in \mathcal{G}_\mathcal{F}$ we have a unique $b_1 \in \widetilde{P}_1$ with $u = R_{b_1} \circ \sigma \circ \pi^1(u)$. We set for $\alpha \in S^1$

$$F^\sigma_\alpha u := R_{b_1} \circ \sigma \circ F_\alpha \circ \pi^1(u).$$

Let $\sigma_2 : \mathcal{CO}(\mathcal{F}) \longrightarrow \mathcal{G}_\mathcal{F}$ be another Weyl structure. Then we have a map $g : \mathcal{CO}(\mathcal{F}) \longrightarrow \widetilde{P}_1$ transfering one Weyl structure into the other, $\sigma = R_g \circ \sigma_2$. Hence for a point $u \in \mathcal{G}_\mathcal{F}$ we can write $u = R_{b_1} \circ \sigma \circ \pi^1(u) = R_{gb_1} \circ \sigma_2 \circ \pi^1(u)$. Consequently we obtain

$$\begin{aligned} F^{\sigma_2}_\alpha u &= R_{gb_1} \circ \sigma_2 \circ F_\alpha \circ \pi^1(u) \\ &= R_{b_1} \circ \underbrace{R_g \circ \sigma_2}_{=\sigma} \circ F_\alpha \circ \pi^1(u) \\ &= F^\sigma_\alpha u. \end{aligned}$$

So the definition of the right-action of S^1 on $\mathcal{G}_\mathcal{F}$ is actually independent of the chosen Weyl structure and we simply write

$$F_\alpha u = F^\sigma_\alpha u \text{ for some Weyl structure } \sigma.$$

Again we have $\pi \circ F_\alpha = f_\alpha \circ \pi$.
According to the definition this S^1-action commutes with the right action of $\widetilde{P}_1$ on the Cartan bundle $\mathcal{G}_\mathcal{F}$.
Furthermore the S^1-action on the Cartan bundle $\mathcal{G}_\mathcal{F}$ inherits the attribute of being free from the S^1-action on the bundle of all conformal frames $\mathcal{CO}(\mathcal{F})$.
By fixing the Weyl structure $\sigma^\theta : \mathcal{CO}(\mathcal{F}) \ni s_\varphi \mapsto ker\big(A^{h_\theta}_{s_\varphi}\big) \in \mathcal{G}_\mathcal{F}$ we obtain the following formula for the Cartan connection (see for example [Feh05])

$$(\sigma^\theta)^*\omega_\mathcal{F} = \theta_{\mathcal{CO}(\mathcal{F})} \oplus A^{h_\theta} \oplus P^{h_\theta}.$$

Here $\theta_{\mathcal{CO}(\mathcal{F})}$ is the displacement form, A^{h_θ} is the Levi Civita connection and P^{h_θ} is the Schouten tensor with respect to the metric h_θ.
We will now study the behaviour of the components of $(\sigma^{h_\theta})^*\omega_\mathcal{F}$ under the action of S^1.
Let γ be a curve in the conformal repere bundle $\mathcal{CO}(\mathcal{F})$.

$$\begin{aligned} (F^*_\alpha \theta_{\mathcal{CO}(\mathcal{F})})_\gamma(\dot\gamma) &= (\theta_{\mathcal{CO}(\mathcal{F})})_{F_\alpha\gamma}(dF_\alpha \circ \dot\gamma) \\ &= [F_\alpha\gamma]^{-1}(d\pi \circ dF_\alpha \circ \dot\gamma) \\ &= [F_\alpha\gamma]^{-1}(F_\alpha \circ d\pi \circ \dot\gamma) \\ &= [\gamma]^{-1}(d\pi \circ \dot\gamma) \\ &= (\theta_{\mathcal{CO}(\mathcal{F})})_\gamma(\dot\gamma) \end{aligned}$$

Let s be a local section of $\mathcal{CO}(\mathcal{F})$ with $\gamma = s \circ \pi \circ \gamma$. We have

$$\begin{aligned} A^\theta_\gamma(\dot\gamma) &= (s^*A^\theta)_{\pi\circ\gamma}(d\pi \circ \dot\gamma) \\ &= \big([s]^{-1}\nabla_{d\pi\circ\dot\gamma}(s_1 \circ \pi \circ \gamma), \ldots, [s]^{-1}\nabla_{d\pi\circ\dot\gamma}(s_{2n+2} \circ \pi \circ \gamma)\big). \end{aligned}$$

Then $\widetilde{s} := F_\alpha \circ s \circ f_{\alpha^{-1}}$ is a local section with $F_\alpha \circ \gamma = \widetilde{s} \circ \pi \circ F_\alpha \circ \gamma$ and we get

$$\begin{aligned} (F^*_\alpha A^\theta)_\gamma(\dot\gamma) &= (\widetilde{s}^*A^\theta)_{\pi\circ F_\alpha\circ\gamma}(d\pi \circ dF_\alpha \circ \dot\gamma) \\ &= \big([\widetilde{s}]^{-1}\nabla_{d\pi\circ dF_\alpha\circ\dot\gamma}(\widetilde{s}_1 \circ \pi \circ F_\alpha \circ \gamma), \ldots, [\widetilde{s}]^{-1}\nabla_{d\pi\circ dF_\alpha\circ\dot\gamma}(\widetilde{s}_{2n+2} \circ \pi \circ F_\alpha \circ \gamma)\big) \\ &= \big([F_\alpha \circ s \circ f_{\alpha^{-1}}]^{-1}\nabla_{F_\alpha\circ d\pi\circ\dot\gamma}(F_\alpha \circ s_1 \circ \pi \circ \gamma), \ldots \\ &\qquad \ldots, [F_\alpha \circ s \circ f_{\alpha^{-1}}]^{-1}\nabla_{F_\alpha\circ d\pi\circ\dot\gamma}(F_\alpha \circ s_{2n+2} \circ \pi \circ \gamma)\big) \\ &= \big([F_\alpha \circ s \circ f_{\alpha^{-1}}]^{-1}F_\alpha\nabla_{d\pi\circ\dot\gamma}(\circ s_1 \circ \pi \circ \gamma), \ldots \\ &\qquad \ldots, [F_\alpha \circ s \circ f_{\alpha^{-1}}]^{-1}F_\alpha\nabla_{d\pi\circ\dot\gamma}(s_{2n+2} \circ \pi \circ \gamma)\big) \\ &= \big([s]^{-1}\nabla_{d\pi\circ\dot\gamma}(\circ s_1 \circ \pi \circ \gamma), \ldots, [s]^{-1}\nabla_{d\pi\circ\dot\gamma}(s_{2n+2} \circ \pi \circ \gamma)\big) \\ &= A^\theta_\gamma(\dot\gamma). \end{aligned}$$

For the third component we have $(\sigma^\theta)^*\omega_1(\dot\gamma) = \sum_j P(d\pi\dot\gamma, \gamma_j)e_j^t$ and therefore

$$\begin{array}{rcl}\big(F_\alpha^*(\sigma^\theta)^*\omega_1\big)_\gamma(\dot\gamma) &=& \sum_j P\big(d\pi \circ dF_\alpha \circ \dot\gamma, (F_\alpha \circ \gamma)_j\big)e_j^t \\ &=& \sum_j P\big(\underbrace{F_\alpha}_{\text{isometry}} \circ d\pi \circ \dot\gamma, F_\alpha \circ \gamma_j\big)e_j^t \\ &=& \sum_j P(d\pi\dot\gamma, \gamma_j)e_j^t \\ &=& (\sigma^\theta)^*\omega_1(\dot\gamma).\end{array}$$

Hence for all $\alpha \in S^1$ we get $F_\alpha^*(\sigma^\theta)^*\omega_{\mathcal{F}} = (\sigma^\theta)^*\omega_{\mathcal{F}}$. Since furthermore the action of S^1 commutes with the action of $\widetilde{P}_1$ the Cartan connection is invariant under the action F_α,

$$F_\alpha^*\omega_{\mathcal{F}} = \omega_{\mathcal{F}}.$$

This actually implies that the Lie derivative of the Cartan connection in the direction of the fundamental vector field generated by the S^1-action vanishes.

$$\begin{array}{rcl}\mathcal{L}_{\widetilde{i}}\omega_{\mathcal{F}} &=& \frac{d}{dt}F_{e^{it}}^*\omega_{\mathcal{F}}\big|_{t=0} \\ &=& \frac{d}{dt}\omega_{\mathcal{F}}\big|_{t=0} \\ &=& 0\end{array}$$

As we have seen in Section 5.7 the local results also hold for the construction of [CG08] and vice versa. Thus we also have according to Section 5.6 that the curvature $\Omega^{\omega_{\mathcal{F}}}$ vanishes if lifts of $\widetilde{i}$ are inserted.

7.1.2 Prolonging the S^1-Action

In the subsection above we have discribed the S^1-action $F : S^1 \times \mathcal{G}_{\mathcal{F}} \longrightarrow \mathcal{G}_{\mathcal{F}}$ on the Cartan bundle of the Fefferman space, which leaves the Cartan connection $\omega_{\mathcal{F}}$ invariant. Hence the S^1-action is an isometry with respect to the metric d_ϱ induced by the Cartan connection, since for any curve $\gamma : I \longrightarrow \mathcal{G}_{\mathcal{F}}$ and any element $\alpha \in S^1$ we have

$$\begin{array}{rcl}\ell\big(F_\alpha\gamma\big) &=& \int_I \big\|\omega_{\mathcal{F}}\big(dF_\alpha\dot\gamma\big)\big\|_{\mathfrak{g}}\,dt \\ &=& \int_I \big\|\underbrace{F_\alpha^*\omega_{\mathcal{F}}}_{=\omega_{\mathcal{F}}}(\dot\gamma)\big\|_{\mathfrak{g}}dt \\ &=& \ell(\gamma).\end{array}$$

So the S^1-action on $\mathcal{G}_{\mathcal{F}}$ can be prolonged to the Cartan boundary $\partial_{CB}\mathcal{G}_{\mathcal{F}}$. Recall that the boundary points are defined by inextendable curves $\gamma : I \longrightarrow \mathcal{G}_{\mathcal{F}}$ which are of finite length $\ell(\gamma) = \int_I \|\omega_{\mathcal{F}}(\dot\gamma)\|_{\widetilde{\mathfrak{g}}}dt < \infty$. We set

$$F_\alpha\big[\gamma\big] := \big[F_\alpha\gamma\big] \text{ for } \big[\gamma\big] \in \partial_{CB}\mathcal{G}_{\mathcal{F}} \text{ and } \alpha \in S^1.$$

Since S^1 acts by isometries on $\mathcal{G}_{\mathcal{F}}$ its action on $\overline{\mathcal{G}_{\mathcal{F}}}$ is continuous.
Now we want to ananlyse the second construction to find more informations on the boundary of the Fefferman space.

7.2 The Construction of [CG08]

If the global $(n+2)$nd root of the canonical line bundle of the CR manifold is given, we can apply the construction from [CG08] as presented in chapter 5 and obtain the following picture.

$$
\begin{tikzcd}
\omega_M & (\omega_{\mathcal{F}})_{[u,p]} = Ad(p^{-1}) \circ (\pi_1^* \omega_M)_{[u,p]} + (\pi_2^* \omega_{\widetilde{P}})_{[u,p]} \\[-2em]
\mathcal{G}_M \arrow[r, hook] \arrow[d, "G \cap \widetilde{P} \circlearrowright"'] \arrow[dd, bend right=60, "{P \circlearrowright = S^1 \ltimes (G \cap \widetilde{P})}"'] & \mathcal{G}_{\mathcal{F}} = \mathcal{G}_M \times_{G \cap \widetilde{P}} \widetilde{P} \arrow[dl, "\circlearrowleft \widetilde{P}"] \\
\mathcal{F} = \mathcal{G}_M /_{G \cap \widetilde{P}} \arrow[d, "S^1 \circlearrowright"'] & \\
M &
\end{tikzcd}
$$

Thus we can consider $\mathcal{G}_M = \mathcal{G}_M \times \{e\} \subset \mathcal{G}_{\mathcal{F}}$ as a Riemannian submanifold since on this submanifold the metrics induced by ω_M respectively $\omega_{\mathcal{F}}$ are identical. And we get the proposition below.

Proposition 7.1 *Let (M, H, θ) be a strictly pseudo-convex CR manifold joined by a complex line bundle $\mathcal{E}(1,0) \longrightarrow M$ together with a duality between $\mathcal{E}(1,0)^{\otimes(n+2)}$ and the canonical complex line bundle $\mathcal{K}$ of M. Then we have*

$$\overline{\mathcal{G}_M \times \{e\}} \subset \overline{\mathcal{G}_{\mathcal{F}}} \text{ and especially } \partial_{CB} M \subset pr(\partial_{CB} \mathcal{F}).$$

In the construction of [CG08] we have identified S^1 as a subgroup of P. So we naturally have a S^1-action on the Cartan bundle $\mathcal{G}_M$ given by the right action of P

$$R_\alpha : \mathcal{G}_M \longrightarrow \mathcal{G}_M.$$

When considering the Adjoint action of S^1 on the $|2|$-graded Lie algebra $\mathfrak{g}$ we find that the norm of the Cartan connection ω_M is invariant under the action of S^1 since with $(e^{it}, e^{\frac{-2it}{n}} In) = \begin{pmatrix} e^{it} & & \\ & e^{-\frac{2it}{n}} I_n & \\ & & e^{it} \end{pmatrix}$ it is

$$
\begin{aligned}
& \left\| Ad\big((e^{it}, e^{\frac{-2it}{n}} In)^{-1}\big)(E_{-2}(a) \oplus E_{-1}(X) \oplus E_0(z, A) \oplus E_1(Z) \oplus E_2(b)) \right\| \\
= & \left\| E_{-2}(a) \oplus E_{-1}(e^{\frac{i(2+n)t}{n}} X) \oplus E_0(z, A) \oplus E_1(e^{\frac{i(2+n)t}{n}} Z) \oplus E_2(b) \right\| \\
= & \left\| E_{-2}(a) \oplus E_{-1}(X) \oplus E_0(z, A) \oplus E_1(Z) \oplus E_2(b) \right\|.
\end{aligned}
$$

I.e. we have $\|R_\alpha^* \omega_M\| = \|\omega_M\|$ for all $\alpha \in S^1$ and therefore the Lie derivative of the norm of the Cartan connection ω_M in direction of the fundamental vector field generated by i vanishes, $\mathcal{L}_{\widetilde{i}} \|\omega_M\| = 0$.
Consequently S^1 acts by isometries on $\mathcal{G}_M$ with respect to the Riemannian metric induced by the Cartan connection and we can prolong this action to a continuous action on $\overline{\mathcal{G}_M}$.

Remark 7.1 *Please note that $\alpha^{-1} \cdot \widetilde{P} \cdot \alpha$ is not always an element of $\widetilde{P}$. Thus an action on the Cartan bundle $\mathcal{G}_{\mathcal{F}}$ like $R_\alpha[u.\widetilde{p}] = [R_\alpha u, \alpha^{-1} \widetilde{p} \alpha]$ is not well defined. We can lift the S^1-action of the Fefferman space in the same way as we have done in Subsection 7.1.1 and obtain the same results. Even if the link between both Cartan bundles and both Cartan connections suggests to compare the S^1-action on $\mathcal{G}_M$ and $\mathcal{G}_{\mathcal{F}}$ we want to point out, that the lifts of the action on $\mathcal{F}$ to $\mathcal{G}_M$ and $\mathcal{G}_{\mathcal{F}}$ are quite different although they project onto the same action on the Fefferman space.*

7.3 Summary

Since local results on the Cartan geometries of a CR manifold and the corresponding Fefferman space do not depend on the construction used we now want to give an owerview on the results which apply for both constructions. Let $(M^{2n+1}, T_{10}, \theta)$ be a strictly pseudo-convex CR manifold and $\mathcal{F}$ the corresponding Fefferman space endowed with an S^1-action. The Cartan geometries of both are denoted by $(\mathcal{G}_M, \pi_M, M; \omega_M)$ and $(\mathcal{G}_\mathcal{F}, \pi_\mathcal{F}, \mathcal{F}; \omega_\mathcal{F})$ respectively. Further $\widetilde{i}$ denotes the fundamental vector field generated by $i \in i\mathbb{R} = LA(S^1)$. Then we have

- S^1 acts by conformal isomorphisms on $\mathcal{F}$.
- The norms of both Cartan connections are invariant under the action of S^1,

$$\mathcal{L}_{\widetilde{i}}\|\omega_M\| = 0 \text{ and } \mathcal{L}_{\widetilde{i}}\|\omega_\mathcal{F}\| = 0.$$

- The curvatures of both Cartan connections vanish if lifts of the fundamental vector field generated by $i \in LA(S^1)$ are inserted,

$$\Omega^{\omega_M}(\widetilde{i}, \cdot) = 0 \text{ and } \Omega^{\omega_\mathcal{F}}(\widetilde{i}, \cdot) = 0.$$

- S^1 acts by isometries on $\mathcal{G}_M$ and $\mathcal{G}_\mathcal{F}$ with respect to the Riemannian metrics induced by the Cartan connections.
- The S^1 actions on $\mathcal{G}_M$ and $\mathcal{G}_\mathcal{F}$ can be prolonged to the boundaries and the actions $S^1 \times \overline{\mathcal{G}_M} \longrightarrow \overline{\mathcal{G}_M}$ and $S^1 \times \overline{\mathcal{G}_\mathcal{F}} \longrightarrow \overline{\mathcal{G}_\mathcal{F}}$ are continuous.
- For the construction of [CG08] we have $\partial_{CB}M \subset pr(\partial_{CB}\mathcal{F})$.

7.4 The Cartan Boundary of the Homogeneous CR-Manifold and its Fefferman Space

The constructions of the homogeneous CR manifold and its Fefferman space have been discussed in Sections 5.1 and 5.2. Denoting $G := SU(p+1, q+1)$ and $P := Stab(\mathbb{C} \cdot \widetilde{\ell})$ where $\widetilde{\ell}$ is a real null line in $\mathbb{C}^{p+1,q+1}$ the homogeneous model of CR-geometry is given by $G/_P$. Its Fefferman space is the Möbius space $\widetilde{G}/_{\widetilde{P}}$ with $\widetilde{G} = O_c(\mathbb{C}^{p+1,q+1}, \langle\cdot,\cdot\rangle_\mathbb{R}) = O_c(2p+2, 2q+2)$ and $\widetilde{P}$ is the stabiliser in $\widetilde{G}$ of the real null line $\widetilde{\ell}$.

$$\begin{array}{cc} & \widetilde{G}/_{\widetilde{P}} \\ S^1 \hookrightarrow & \downarrow \\ & G/_P \end{array}$$

So since both of the spaces are homogeneous their Cartan boundaries are empty as we have discussed in Subsection 6.2.1,

$$\partial_{CB}G/_P = \emptyset \text{ and } \partial_{CB}\widetilde{G}/_{\widetilde{P}} = \emptyset.$$

Chapter 8

The Heisenberg Group

In this chapter we will explicitly compute the Cartan boundaries of the Heisenberg group $He(n)$ and its Fefferman space. As we have discussed in Subsection 2.1.1 the Heisenberg group is a CR manifold. And we will see that die Heisenberg group is flat. So next to the homogeneous space this is one of the basic examples of CR manifolds. However the Heisenberg group is - contrary to the homogeneous space - not compact, which causes us to expect the boundary to be nonempty.

Recall that the Heisenberg group can be realised as

$$He(n) := \left\{ \begin{pmatrix} 1 & X^t & z \\ 0 & I_n & Y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} z \in \mathbb{R} \\ X, Y \in \mathbb{R}^n \end{array} \right\} \subset Gl(n+2)$$

with the corresponding Lie algebra given by:

$$\mathfrak{he}(n) := LA(He(n)) = \left\{ \begin{pmatrix} 0 & X^t & z \\ 0 & 0 & Y \\ 0 & 0 & 0 \end{pmatrix} =: M(X,Y,z) \;\middle|\; \begin{array}{l} z \in \mathbb{R} \\ X, Y \in \mathbb{R}^n \end{array} \right\}.$$

Then a basis of the Lie algebra $\mathfrak{he}(n)$ is given by the tuple $\big(X_i := M(e_i,0,0), Y_i := M(0,e_i,0), Z := M(0,0,1) \mid i = 1,\dots,n\big)$ and we have for the Lie brackets $[X_i, Y_i] = Z$ for $i = 1,\dots,n$ and the remaining brackets vanish. Thus the Heisenberg algebra $\mathfrak{he}(n)$ is nilpotent of order two. Further the Baker-Campbell-Hausdorff formula reduces to

$$\exp(A) \cdot \exp(B) = \exp\big(A + B + \frac{1}{2}[A,B]_{\mathfrak{he}(n)}\big).$$

The exponential map is a global diffeomorphism.

$$\begin{array}{rcl} \exp : \mathfrak{he}(n) & \longrightarrow & He(n) \\ M(X,Y,z) & \mapsto & \exp\big(M(X,Y,z)\big) \\ & & = \sum_{i=0}^{\infty} \frac{1}{i!} \begin{pmatrix} 0 & X^t & z \\ 0 & 0 & Y \\ 0 & 0 & 0 \end{pmatrix}^i \\ & & = I_{n+2} + \begin{pmatrix} 0 & X^t & z \\ 0 & 0 & Y \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & X^tY \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ & = & \begin{pmatrix} 1 & X^t & z + X^tY \\ 0 & I_n & Y \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

As we have seen in Subsection 2.1.1 the subbundle H is spanned by the left invariant vector fields defined by X_i and Y_i, $i = 1, \dots n$. And a global frame is given by the left invariant vector fields $(\widetilde{X_1}, \dots, \widetilde{X_n}, \widetilde{Y_1}, \dots, \widetilde{Y_n}, \widetilde{Z})$ given in the usual way

$$\widetilde{A}(u) := \left. \frac{d}{dt}\Big(R_{\exp(tA)} u \Big) \right|_{t+0}.$$

Please note that $\widetilde{Z} = T$ is the Reeb vector field of the pseudo-hermitian form $\theta = \widetilde{Z}^*$.

8.1 The Levi Form and the Tanaka Webster Connection

Computing the commutator of such vector fields we obtain for any linear function f:

$$\begin{aligned}
[\widetilde{A},\widetilde{B}](f)_u &= \widetilde{A}\big(\widetilde{B}(f)\big)_u - \widetilde{B}\big(\widetilde{A}(f)\big)_u \\
&= \widetilde{A}\left(\tfrac{d}{dt} f(R_{\exp(tB)}u)\big|_{t=0}\right) - \widetilde{B}\left(\tfrac{d}{dt} f(R_{\exp(tA)}u)\big|_{t=0}\right) \\
&= \tfrac{d}{ds}\ \tfrac{d}{dt} f(R_{\exp(sA)} \circ R_{\exp(tB)}u)\big|_{t=0,s=0} - \tfrac{d}{ds}\ \tfrac{d}{dt} f(R_{\exp(sB)} \circ R_{\exp(tA)}u)\big|_{t=0,s=0} \\
&= \tfrac{d}{ds}\tfrac{d}{dt} f\left((R_{\exp(tB)\cdot\exp(sA)} - R_{\exp(tA)\cdot\exp(sB)})u\right)\big|_{t=0,s=0} \\
&= \tfrac{d}{ds}\tfrac{d}{dt} f\left(R_{\exp(tB+sA+\frac{1}{2}[tB,sA]_{\mathfrak{he}(n)})-\exp(tA+sB+\frac{1}{2}[tA,sB]_{\mathfrak{he}(n)})}u\right)\Big|_{t=0,s=0}.
\end{aligned}$$

If we insert for A and B the basis vectors $(X_1,\ldots,X_n,Y_1,\ldots,Y_n,Z)$ the Lie bracket vanishes in all cases exept for $[X_i,Y_i]_{\mathfrak{he}(n)} = Z$. If the Lie bracket is zero the remaining terms vanish as well and the commutator of the vector fields is zero. For $A = X_i$ and $B = Y_i$ we have

$$\begin{aligned}
[\widetilde{X_i},\widetilde{Y_i}](f)_u &= \tfrac{d}{ds}\tfrac{d}{dt} f\left(R_{\exp(tY_i+sX_i-\frac{st}{2}Z)-\exp(tX_i+sY_i+\frac{st}{2}Z)}u\right)\Big|_{t=0,s=0} \\
&= \tfrac{d}{ds}\tfrac{d}{dt} f\big(R_{M(0,0,-st)}u\big)\big|_{t=0,s=0} \\
&= -\widetilde{Z}(f)_u \\
&= -\widetilde{[X_i,Y_i]}_{\mathfrak{he}(n)}(f)_u.
\end{aligned}$$

I.e. for $A,B \in \{X_1,\ldots,X_n,Y_1,\ldots,Y_n,Z\}$ we get $[\widetilde{A},\widetilde{B}] = -\widetilde{[A,B]}_{\mathfrak{he}(n)}$.
With this we will now compute the Levi form G_θ as defined in Subsection 2.1.2.

$$\begin{aligned}
G_\theta\big(\widetilde{X}_i,\widetilde{Y}_j\big) &= d\theta\big(\widetilde{X}_i,J\widetilde{Y}_j\big) \\
&= d\theta\big(\widetilde{X}_i,-\widetilde{X}_j\big) \\
&= -\theta\underbrace{\big[\widetilde{X}_i,-\widetilde{X}_j\big]}_{=0} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
G_\theta\big(\widetilde{X}_i,\widetilde{X}_j\big) &= -\theta\big[\widetilde{X}_i,J\widetilde{X}_j\big] \\
&= -\theta\quad\underbrace{\big[\widetilde{X}_i,\widetilde{Y}_j\big]}_{=-\widetilde{[X_i,Y_j]}_{\mathfrak{he}(n)}} \\
&= \theta(\delta_{ij}\widetilde{Z}) \\
&= \delta_{ij} \text{ and}
\end{aligned}$$

$$\begin{aligned}
G_\theta\big(\widetilde{Y}_i,\widetilde{Y}_j\big) &= -\theta\big[\widetilde{Y}_i,J\widetilde{Y}_j\big] \\
&= -\theta\big[\widetilde{Y}_i,-\widetilde{X}_j\big] \\
&= \delta_{ij}.
\end{aligned}$$

Thus with respect to the Webster metric $g_\theta := G_\theta + \theta \odot \theta$ the global frame $(\widetilde{X}_1,\ldots,\widetilde{X}_n,\widetilde{Y}_1,\ldots,\widetilde{Y}_n,\widetilde{Z})$ is orthonormal and g_θ is positiv definite. I.e. the Heisenberg group $(He(n),\theta)$ is strictly pseudo-convex.
Now we are going to compute the Tanaka Webster connection as defined in Subsection 2.1.4. The subbundle $T_{10} \subset TM^{\mathbb{C}}$ is given as

$$\begin{aligned}
T_{10} &= span\{\widetilde{X}_j - iJ\widetilde{X}_j, \widetilde{Y}_j - iJ\widetilde{Y}_j \mid j = 1,\ldots,n\} \\
&= span\{\widetilde{X}_j - i\widetilde{Y}_j \mid j = 1,\ldots,n\}.
\end{aligned}$$

For sections $X, Y \in \Gamma(T_{10})$ the Tanaka Webster connection is defined via

- $\nabla^W_T X := pr_{T_{10}}[T, X]$, i.e.

$$\begin{aligned} \nabla^W_T(\widetilde{X}_j - i\widetilde{Y}_j) &= pr_{T_{10}}[T, \widetilde{X_j - iY_j}] \\ &= -pr_{T_{10}}[Z, X_j - iY_j]_{\mathfrak{he}(n)} \\ &= 0. \end{aligned}$$

- $\nabla^W_{\overline{Y}} X := pr_{T_{10}}[\overline{Y}, X]$ that is to say

$$\begin{aligned} \nabla^W_{\overline{\widetilde{X}_k - i\widetilde{Y}_k}}(\widetilde{X}_j - i\widetilde{Y}_j) &= pr_{T_{10}} \underbrace{[\widetilde{X}_k + i\widetilde{Y}_k, \widetilde{X}_j - i\widetilde{Y}_j]}_{\in span\{T\}} \\ &= 0. \end{aligned}$$

- $L_\theta(\nabla^W_Y X, Z) \stackrel{!}{=} Y\big(L_\theta(X, Z)\big) - L_\theta\big(X, [\overline{Y}, Z]\big)$ for all sections $Z \in \Gamma(T_{10})$, i.e.

$$\begin{aligned} & L_\theta\left(\nabla^W_{\widetilde{X}_k - i\widetilde{Y}_k}(\widetilde{X}_j - i\widetilde{Y}_j), \widetilde{X}_l - i\widetilde{Y}_l\right) \\ \stackrel{!}{=}\ & (\widetilde{X}_k - i\widetilde{Y}_k)\big(\underbrace{L_\theta(\widetilde{X}_j - i\widetilde{Y}_j, \widetilde{X}_l - i\widetilde{Y}_l)}_{=2\delta_{jl}=const}\big) - L_\theta\big(\widetilde{X}_j - i\widetilde{Y}_j, \underbrace{[\widetilde{X}_k + i\widetilde{Y}_k, \widetilde{X}_l - i\widetilde{Y}_l]}_{\in span\{T\}}\big) \\ =\ & 0. \end{aligned}$$

So according to the extension of ∇^W defined in Subsection 2.1.4 we obtain for all vector fields $A, B \in \{\widetilde{X}_1, \ldots, \widetilde{X}_n, \widetilde{Y}_1, \ldots, \widetilde{Y}_n, \widetilde{Z}\}$ of our global frame

$$\nabla^W_A B = 0.$$

With Proposition 2.2 we obtain for the torsion of the Tanaka Webster connection and sections $X, Y \in \Gamma(H)$

- $Tor^W(X, Y) = G_\theta(JX, Y)T$, that is

$$\begin{aligned} Tor^W(\widetilde{X}_i, \widetilde{Y}_j) &= G_\theta(\underbrace{J\widetilde{X}_i}_{=\widetilde{Y}_i}, \widetilde{Y}_j) \cdot T \\ &= \delta_{ij} T, \\ Tor^W(\widetilde{X}_i, \widetilde{X}_j) &= 0 \\ \text{and} \quad Tor^W(\widetilde{Y}_i, \widetilde{Y}_j) &= 0. \end{aligned}$$

- $Tor^W(T, X) = -\frac{1}{2}\big([T, X] + J[T, JX]\big)$, that is to say

$$\begin{aligned} Tor^W(T, \widetilde{X}_j) &= -\tfrac{1}{2}\big(\underbrace{[T, \widetilde{X}_j]}_{=0} + J\underbrace{[T, J\widetilde{X}_j]}_{=0}\big) \\ &= 0 \\ \text{and in the same way} \quad Tor^W(T, \widetilde{Y}_j) &= 0. \end{aligned}$$

Since the Tanaka Webster connection applied to the vector fields of the global basis vanishes the curvature tensor

$$\mathcal{R}^W(X, Y, Z, W) := g_\theta\big((\nabla^W_X \nabla^W_Y - \nabla^W_Y \nabla^W_X - \nabla^W_{[X,Y]})Z, \overline{W}\big) \quad \text{for } X, Y, Z, W \in \Gamma(TM^{\mathbb{C}})$$

vanishes completely. Thus the Heisenberg group $(He(n), \nabla^W)$ is flat.

8.2 The Cartan Boundary of the Heisenberg Group

Now we use Section 4.4 to calculate the Cartan boundary of the Heisenberg group. The Lie algebra $\mathfrak{g}$ is the special unitary Lie algebra $\mathfrak{g} = \mathfrak{su}(1, 2n+1)$. The subbundle H is spanned by the left invariant vector fields $\widetilde{X}_1, \widetilde{Y}_1, \ldots, \widetilde{X}_n, \widetilde{Y}_n$ and $(\widetilde{X}_1, \ldots, \widetilde{X}_n)$ is a complex global frame of H. The totally real, nondegenerate, bilinear pairing is given by the Levi form $\{\cdot,\cdot\} := G(\cdot, J\cdot)$, which again is defined with the help of the Lie bracket.

$$\begin{aligned}
\{\widetilde{X}_j, \widetilde{Y}_j\} &= G(\widetilde{X}_j, J\widetilde{Y}_j) \\
&= -[\widetilde{X}_j, J^2\widetilde{Y}_j]_{\mathfrak{he}(n)} + H \\
&= \widetilde{[X_j, Y_j]}_{\mathfrak{he}(n)} + H \\
&= \widetilde{Z} + H
\end{aligned}$$

and all other brakets vanish.

8.2.1 The G_0-Bundle

The G_0-principal bundle $p^1 : E^1 \longrightarrow M$ according to Proposition 4.3 which satisfies the structure equations and is harmonic is given as

$$E^1 := \left\{ (\varphi_1, \varphi_2)_x \;\middle|\; \begin{array}{c} x \in M,\ \varphi_1 : \mathfrak{g}_{-1} \longrightarrow T_x^{-1}M \text{ complex-linear isomorphism,} \\ \varphi_2 : \mathfrak{g}_{-2} \longrightarrow T_xM/T_x^{-1}M \text{ linear isomorphism with} \\ \{\varphi_1(X), \varphi_1(Y)\} = \varphi_2[X,Y] \text{ for all } X, Y \in \mathfrak{g}_{-1} \end{array} \right\}.$$

We can describe the complex-linear isomorphism φ_1 by a matrix $A \in Gl(n, \mathbb{C})$ using the global complex frame $(\widetilde{X}_1, \ldots, \widetilde{X}_n)$ for H and $(E_{-1}(e_1), \ldots E_{-1}(e_n))$ for $\mathfrak{g}_{-1}$ (see Section 4.2). The linear isomorphism φ_2 can be described by $\lambda \in \mathbb{R}^*$ with the help of the global frame $\widetilde{Z} + H$ and $E_{-2}(-2) \in \mathfrak{g}_{-2}$. We choose $E_{-2}(-2)$ because of the correspondence $\{\widetilde{X}_j, \widetilde{Y}_j\} = \widetilde{Z} + h$ and $[E_{-1}(e_j), E_{-1}(ie_j)]_{\mathfrak{g}} = E_{-2}(-2)$. Thus we require for (φ_1, φ_2) resp. (A, λ) and all indicies $j, k = 1, \ldots, n$

$$\begin{aligned}
&\quad 0 + H \\
&= \varphi_2\big(\underbrace{[E_{-1}(e_j), E_{-1}(e_k)]_{\mathfrak{g}}}_{=0}\big) \overset{!}{=} \big\{\varphi_1(E_{-1}(e_j)), \varphi_1(E_{-1}(e_k))\big\} \\
&= \big\{\textstyle\sum_{l=1}^n A_{lj}\widetilde{X}_l, \sum_{m=1}^n A_{mk}\widetilde{X}_m\big\} \\
&= \textstyle\sum_{l=1}^n \big(Re(A_{lj})Im(A_{lk}) - Im(A_{lj})Re(A_{lk})\big)\widetilde{Z} + H \\
&= Im(A^*A)_{jk}\widetilde{Z} + H
\end{aligned}$$

and

$$\begin{aligned}
&\quad \lambda\delta_{jk}\widetilde{Z} + H \\
&= \varphi_2\big(\underbrace{[E_{-1}(e_j), E_{-1}(ie_k)}\ ]_{\mathfrak{g}}\big) \overset{!}{=} \big\{\varphi_1(E_{-1}(e_j)), \varphi_1(E_{-1}(ie_k))\big\} \\
&\qquad = E_{-2}\big(2Im((ie_k)^*e_j\big) \\
&\qquad = \delta_{jk}E_{-2}(-2)\big) \\
&= \big\{\textstyle\sum_{l=1}^n A_{lj}\widetilde{X}_l, \sum_{m=1}^n A_{mk}J\widetilde{X}_m\big\} \\
&= \textstyle\sum_{l=1}^n \big(Re(A_{lj})Re(A_{lk}) + Im(A_{lj})Im(A_{lk})\big)\widetilde{Z} + H \\
&= Re(A^*A)_{jk}\widetilde{Z} + H.
\end{aligned}$$

I.e. $A^*A \overset{!}{=} \lambda I_n$. We can choose $B \in U(n)$ and $a \in \mathbb{C}^*$ such that $A = a^{-1}B$, $\lambda = |a|^{-2}$ and $a\overline{a}^{-1}det(B) = 1$. However this choice is only unique up to $\mathbb{Z}_{n+2}$ since

$$A = a^{-1}B = (e^{i\varphi}a)^{-1}(e^{i\varphi}B) \text{ and } 1 \overset{!}{=} (e^{i\varphi}a)(\overline{e^{i\varphi}a})^{-1}det(e^{i\varphi}B) = (e^{i\varphi})^{(n+2)}a\overline{a}^{-1}B.$$

By this choice (φ_1,φ_2) actually corresponds to the Adjoint action of $(a,B)=\begin{pmatrix} a & & \\ & B & \\ & & \overline{a}^{-1}\end{pmatrix}$ followed by the pointwise isomorphism

$$
\begin{array}{rcl}
\Phi_x : \mathfrak{g}_{-1}\oplus\mathfrak{g}_{-2} & \longrightarrow & T_{(X,Y,z)}He(n) \\
E_{-1}(e_j) & \mapsto & (\widetilde{X}_j)_{(X,Y,z)} = dL_{(X,Y,z)}\begin{pmatrix} 0 & e_j^t & 0 \\ & & 0 \\ & & 0\end{pmatrix} = \begin{pmatrix} 0 & e_j^t & 0 \\ & & 0 \\ & & 0\end{pmatrix}, \\
E_{-1}(ie_j) & \mapsto & (\widetilde{Y}_j)_{(X,Y,z)} = dL_{(X,Y,z)}\begin{pmatrix} 0 & 0 & 0 \\ & & e_j \\ & & 0\end{pmatrix} = \begin{pmatrix} 0 & 0 & x_j \\ & & e_j \\ & & 0\end{pmatrix}, \\
E_{-2}(-2) & \mapsto & (\widetilde{Z})_{(X,Y,z)} = dL_{(X,Y,z)}\begin{pmatrix} 0 & 0 & 1 \\ & & 0 \\ & & 0\end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ & & 0 \\ & & 0\end{pmatrix}.
\end{array}
$$

I.e.

$$
\begin{array}{rcl}
(\varphi_1)_x\big(E_{-1}(e_j)\big) & = & \Phi_x \circ Ad(a,B)E_{-1}(e_j) \\
& = & \Phi_x\left(\sum_k a^{-1}B_{kj}E_{-1}(e_k)\right) \\
& = & \sum_k a^{-1}B_{kj}\widetilde{X}_k, \\
(\varphi_1)_x\big(E_{-1}(ie_j)\big) & = & \sum_k a^{-1}B_{kj}\widetilde{Y}_k \text{ and} \\
(\varphi_2)_x\big(E_{-2}(-2)\big) & = & \Phi_x \circ Ad(a,B)E_{-2}(-2) \\
& = & \Phi_x\left(E_{-2}(\frac{-2}{|a|^2})\right) \\
& = & \frac{1}{|a|^2}\widetilde{Z} + H.
\end{array}
$$

Thus the right action of $\underline{G_0}$ on E^1 as defined in Section 4.4 via composition with the Adjoint action is actually the right action of the group $\underline{G_0}$ on itself.

$$
R_{(\alpha,A)}(\varphi_1,\varphi_2) = (\varphi_1,\varphi_2)\circ Ad(\alpha,A) = \Phi\circ Ad(a,B)\circ Ad(\alpha,A) = \Phi\circ Ad\big(R_{(\alpha,A)}(a,B)\big)
$$

Consequently the bundle E^1 is just the trivial bundle

$$
E^1 = He(n)\times G_0/_{\mathbb{Z}_{n+2}} = He(n)\times\underline{G_0} \xrightarrow{p^1} He(n)
$$

with $\underline{G_0}$ acting on the second factor from the right.

The frame form θ^1 of lenth one was defined by

$$
\begin{array}{rcll}
\theta^1_{-2}(\varphi)(\xi) & := & \varphi_2^{-1}([dp^1\xi]) & \text{for all } \xi\in T_\varphi E^1 = T_\varphi^{-2}E^1 \text{ and} \\
\theta^1_{-1}(\varphi)(\xi) & := & \varphi_1^{-1}(\underbrace{dp^1\xi}_{\in T_x^{-1}M}) & \text{for all } \xi\in d(p^1)^{-1}(T_x^{-1}M) = T_\varphi^{-1}E^1.
\end{array}
$$

Writing for the tangent vector $\xi = \overbrace{\begin{pmatrix} 0 & \dot{X}^t & \dot{z} \\ & 0 & \dot{Y} \\ & & 0 \end{pmatrix}}^{\in T_{(X,Y,z)}He(n)} + \xi_{\underline{G_0}}$ the frame form θ^1 at the point $\varphi = (\varphi_1, \varphi_2)_{(X,Y,z)} = \big((X,Y,z),(a,B)\big)$ gives

$$\begin{aligned}
(\theta^1_{-2})_\varphi(\xi) &= \varphi_2^{-1}\left[\begin{pmatrix} 0 & \dot{X}^t & \dot{z} \\ & 0 & \dot{Y} \\ & & 0 \end{pmatrix}\right] \\
&= \varphi_2^{-1}\Big[\textstyle\sum_j \dot{x}_j \widetilde{X}_j + \sum_j \dot{y}_j \widetilde{Y}_j + (\dot{z} - \sum_j x_j \dot{y}_j)\widetilde{Z}\Big] \\
&= \varphi_2^{-1}\big((\dot{z} - \textstyle\sum_j x_j \dot{y}_j)\widetilde{Z} + H\big) \\
&= Ad(a^{-1}, B^{-1}) \circ \Phi_x^{-1}\big((\dot{z} - \textstyle\sum_j x_j \dot{y}_j)\widetilde{Z} + H\big) \\
&= Ad(a^{-1}, B^{-1})\big(E_{-2}\big(-2(\dot{z} - \textstyle\sum_j x_j \dot{y}_j)\big)\big) \\
&= E_{-2}\big(-2|a|^2(\dot{z} - \textstyle\sum_j x_j \dot{y}_j)\big) \text{ and}
\end{aligned}$$

$$\begin{aligned}
(\theta^1_{-1})_\varphi\big(\underbrace{\xi}_{\in T^{-1}_\varphi E^1}\big) &= \varphi_1^{-1}\big(\underbrace{\begin{pmatrix} 0 & \dot{X}^t & \sum_j x_j \dot{y}_j \\ & 0 & \dot{Y} \\ & & 0 \end{pmatrix}}_{\in H}\big) \\
&= Ad(a^{-1}, B^{-1}) \circ \Phi_x^{-1}\big(\textstyle\sum_{j=1}^n \dot{x}_j \widetilde{X}_j + \dot{y}_j \widetilde{Y}_j\big) \\
&= Ad(a^{-1}, B^{-1})\big(\textstyle\sum_{j=1}^n (\dot{x}_j + i\dot{y}_j)_j E_{-1}(e_j)\big) \\
&= Ad(a^{-1}, B^{-1})\big(E_{-1}(\dot{X} + i\dot{Y})\big) \\
&= E_{-1}\big(aB^{-1}(\dot{X} + i\dot{Y})\big).
\end{aligned}$$

According to Section 4.4 the bundle $(E^1, p^1, He(n); \theta^1)$ is a harmonic $\underline{P}$-frame bundle of degree one.

8.2.2 The First Prolongation

The bundle E^1 needs to be prolonged as explained in Section 4.4. Thus we consider

$$\hat{E} := \left\{ \hat{\varphi} = (\hat{\varphi}_{-2}, \hat{\varphi}_{-1}) \;\middle|\; \begin{array}{ll} \hat{\varphi}_{-2}: & T^{-2}E^1 \longrightarrow \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \text{ an isomorphism with} \\ & pr_{\mathfrak{g}_{-2}} \circ \hat{\varphi}_{-2} = \theta^1_{-2} \text{ and } \hat{\varphi}_{-2}|_{T^{-1}E^1} = \theta^1_{-1} \\ \hat{\varphi}_{-1}: & T^{-1}E^1 \longrightarrow \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \text{ an isomorphism with} \\ & pr_{\mathfrak{g}_{-1}} \circ \hat{\varphi}_{-1} = \theta^1_{-1} \text{ and } \hat{\varphi}_{-1}(\widetilde{A}) = A \;\text{ for all } A \in \mathfrak{g}_0 \end{array} \right\}.$$

A global section of this bundle, $\pi : \hat{E} \longrightarrow E^1$, is given by

$$\begin{aligned}
\sigma : E^1 &\longrightarrow \hat{E} \\
(\varphi_1, \varphi_2) &\mapsto \big(\sigma(\varphi)_{-2}, \sigma(\varphi)_{-1}\big) := \big(\theta^1_{-2} \oplus \theta^1_{-1}, \theta^1_{-1} \oplus \omega_{\underline{G_0}}\big).
\end{aligned}$$

As discussed in Section 4.4 the prolonged bundle E^2 is the bundle of all elements of $\hat{E}$ with ∂^*-closed homogeneous component of degree one of the torsion. Thus we need to study the torsion and therefore $\hat{\theta}$, the analog of the frame form. This is defined as $(\hat{\theta}_j)_{\hat{\varphi}} := \hat{\varphi}_j \circ d\pi$. The torsion of an element $\hat{\varphi} \in \hat{E}$ is defined with the help of a section σ for $\dot{X} \in \mathfrak{g}_i$, $\dot{Y} \in \mathfrak{g}_j$ and $\xi \in T^i_{\pi\hat{\varphi}}E^1$, $\eta \in T^j_{\pi\hat{\varphi}}E^1$ $(i, j < 0)$ with $\hat{\varphi}_i(\xi) = \dot{X}$ and $\hat{\varphi}_j(\eta) = \dot{Y}$ via

$$\begin{aligned}
t_{\hat{\varphi}}(\dot{X}, \dot{Y}) &:= \begin{cases} pr_{\mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_{i+j+1}} \circ d(\sigma^*\hat{\theta}_{-2})_{\pi\hat{\varphi}}(\xi, \eta), & i+j < -2 \\ d(\sigma^*\hat{\theta}_{i+j})_{\pi\hat{\varphi}}(\xi, \eta), & i+j \geq -2 \end{cases} \\
&\in \mathfrak{g}_{i+j} \oplus \mathfrak{g}_{i+j+1}.
\end{aligned}$$

For $\varphi = \big((X,Y,z),(a,B)\big) \in E^1$, $\dot{X} = E_{-2}(\dot{X}) \in \mathfrak{g}_{-2}$ and $\xi \in T^{-2}_{\varphi}E^1$ with $\sigma(\varphi)_{-2}(\xi) = \dot{X}$ the vector $\xi = \begin{pmatrix} 0 & 0 & -\frac{\dot{X}}{2|a|^2} \\ & & 0 \\ & & 0 \end{pmatrix} \in T^{-2}_{\varphi}E^1$ fullfils

$$\begin{aligned}
\mathfrak{g}_{-2} \ni \dot{X} &\overset{!}{=} \sigma(\varphi)_{-2}(\xi) \\
&= (\theta^1_{-2} \oplus \theta^1_{-1})\big(\tfrac{\dot{X}}{2|a|^2}\widetilde{Z}\big) \\
&= \theta^1_{-2}\big(-\tfrac{\dot{X}}{2|a|^2}\widetilde{Z}\big) \\
&= E_{-2}(\dot{X}).
\end{aligned}$$

For $\dot{X} = E_{-1}(\dot{X}) \in \mathfrak{g}_{-1}$ we choose the vector $\xi = \begin{pmatrix} 0 & \xi_X^t & \langle X, \xi_Y\rangle \\ & 0 & \xi_Y \\ & & 0 \end{pmatrix} \in T^{-1}_{\varphi}E^1$ with $\dot{X} = aB^{-1}(\xi_X + i\xi_Y)$. Then we obtain

$$\begin{aligned}
\mathfrak{g}_{-1} \ni \dot{X} &\overset{!}{=} \sigma(\varphi)_{-1}(\xi) \\
&= (\theta^1_{-1} + \omega_{\underline{G_0}})\big(\textstyle\sum(\xi_X)_j\widetilde{X}_j + (\xi_Y)_j\widetilde{Y}_j\big) \\
&= E_{-1}\big(aB^{-1}(\xi_X + i\xi_Y)\big) \\
&= E_{-1}(\dot{X}).
\end{aligned}$$

Now we can compute the torsion of the elements $\sigma(\varphi) = \hat{\varphi}$.

- $\dot{X} \in \mathfrak{g}_{-2}$, $\dot{Y} \in \mathfrak{g}_{-1}$

$$\begin{aligned}
t_{\hat{\varphi}}(\dot{X}, \dot{Y}) &= pr_{\mathfrak{g}_{-2}} \circ d(\sigma^*\hat{\theta}_{-2})_{\varphi}(\xi,\eta) \\
&= pr_{\mathfrak{g}_{-2}}\Big(d\sigma\xi\big(\hat{\theta}_{-2}(d\sigma\eta)\big) - d\sigma\eta\big(\hat{\theta}_{-2}(d\sigma\xi)\big) - \hat{\theta}_{-2}\big([d\sigma\xi, d\sigma\eta]\big)\Big) \\
&= pr_{\mathfrak{g}_{-2}}\big(d\sigma\xi\big(\underbrace{\hat{\varphi}_{-2}(\eta)}_{\equiv Y}\big) - d\sigma\eta\big(\underbrace{\hat{\varphi}_{-2}(\xi)}_{\equiv X}\big) - \hat{\varphi}_{-2}\big([\xi,\eta]\big)\big) \\
&= -pr_{\mathfrak{g}_{-2}} \circ \underbrace{\hat{\varphi}_{-2}}_{=\sigma(\varphi)_{-2}=\theta^1_{-2}\oplus\theta^1_{-1}}\big([\xi,\eta]\big) \\
&= -\theta^1_{-2}\big([\xi,\eta]\big) \\
&= -\theta^1_{-2}\underbrace{\Big(\Big[-\frac{\dot{X}}{2|a|^2}\widetilde{Z}, \sum_{j=1}^{n}(\eta_X)_j\widetilde{X}_j + (\eta_Y)_j\widetilde{Y}_j\Big]\Big)}_{=0} \\
&= 0
\end{aligned}$$

- $\dot{X}, \dot{Y} \in \mathfrak{g}_{-1}$

$$\begin{aligned}
t_{\hat{\varphi}}(\dot{X}, \dot{Y}) &= d(\sigma^*\hat{\theta}_{-2})_{\varphi}(\xi,\eta) \\
&= -\hat{\theta}_{-2} \circ d\sigma\big([\xi,\eta]\big) \\
&= -\hat{\varphi}_{-2}\big([\xi,\eta]\big) \\
&= -\hat{\varphi}_{-2}\big(\big[\textstyle\sum(\xi_X)_j\widetilde{X}_j + (\xi_Y)_j\widetilde{Y}_j, \sum(\eta_X)_l\widetilde{X}_l + (\eta_Y)_l\widetilde{Y}_l\big]\big) \\
&= -\hat{\varphi}_{-2}\big(\textstyle\sum_j\big((\xi_X)_j(\eta_Y)_j - (\xi_Y)_j(\eta_X)_j\big)\widetilde{Z}\big) \\
&= -(\theta^1_{-2} \oplus \theta^1_{-1})\big(Im\langle\xi_X + i\xi_Y, \eta_X + i\eta_Y\rangle\widetilde{Z}\big) \\
&= -\theta^1_{-2}\big(Im\langle\xi_X + i\xi_Y, \eta_X + i\eta_Y\rangle\widetilde{Z}\big) \oplus \theta^1_{-1}(0) \\
&\in \mathfrak{g}_{-2}
\end{aligned}$$

So in this case the homogeneous component of degree one vanishes.

Consequently the homogeneous component of degree one of the torsion of the elements $\sigma(\varphi)$ vanishes and is therefore also ∂^*-closed. I.e.

$$\begin{array}{rcl} \sigma : E^1 & \longrightarrow & E^2 \\ (\varphi_{-2},\varphi_{-1}) & \mapsto & \big(\sigma(\varphi)_{-2},\sigma(\varphi)_{-1}\big) := \big(\theta^1_{-2}\oplus\theta^1_{-1},\theta^1_{-1}\oplus\omega_{\underline{G_0}}\big) \end{array}$$

is actually a section of the prolonged bundle E^2. With σ commuting with the right action of $\underline{G}_0$ the bundle $E^2 = He(n)\times\underline{G}_1$ is trivial with $\underline{G}_1$ acting on the second component from the right.

$$\begin{array}{rcl} E^2 & = & He(n)\times \underline{P}/_{\underline{P^2_+}} = He(n)\times\underline{G}_1 \\ & = & \left\{ \begin{array}{rl|l} & (x, g_0\exp(g_1)) & \\ = & R_{\exp(g_1)}\circ\sigma(x,g_0) & x\in He(n), g_0\in\underline{G_0}, g_1\in\mathfrak{g}_1 \\ = & R_{g_0\exp(g_1)}\circ\sigma(x,id) & \end{array}\right\}. \end{array}$$

We do not need to explicitly compute the frame form of length two here.
$(E^2,p^2,He(n);\theta^2)$ is the $\underline{P}$-frame bundle of degree two.

8.2.3 The Second Prolongation

Again we take the steps outlined in Section 4.4.

$$\hat{E} := \left\{ \hat{\varphi} = (\hat{\varphi}_{-2},\hat{\varphi}_{-1}) \;\middle|\; \begin{array}{ll} \hat{\varphi}_{-2}: & T^{-2}E^2 \longrightarrow \mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}\oplus\mathfrak{g}_0 \text{ an isomorphism with} \\ & pr_{\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}}\circ\hat{\varphi}_{-2} = \theta^2_{-2} \text{ and } \hat{\varphi}_{-2}|_{T^{-1}E^2} = \theta^2_{-1} \\ \hat{\varphi}_{-1}: & T^{-1}E^2 \longrightarrow \mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1 \text{ an isomorphism with} \\ & pr_{\mathfrak{g}_{-1}\oplus\mathfrak{g}_0}\circ\hat{\varphi}_{-1} = \theta^2_{-1} \text{ and } \hat{\varphi}_{-1}(\widetilde{A}) = A \;\text{ for all } A\in\mathfrak{g}_0\oplus\mathfrak{g}_1 \end{array}\right\}$$

Now a global section of the new bundle, $\pi:\hat{E}\longrightarrow E^2$, is given by

$$\begin{array}{rcl} \sigma : E^2 & \longrightarrow & \hat{E} \\ (\varphi_{-2},\varphi_{-1}) & \mapsto & \big(\sigma(\varphi)_{-2},\sigma(\varphi)_{-1}\big) := \big(\theta^1_{-2}\oplus\theta^1_{-1}\oplus\omega_{\underline{G_0}},\theta^1_{-1}\oplus\omega_{\underline{P}/_{\underline{P^2_+}}}\big). \end{array}$$

With the same data as before the torsion is defined. For $\dot{X} = E_{-2}(\dot{(X)})\in\mathfrak{g}_{-2}$ we choose the vector field $\xi = -\frac{\dot{X}}{2|a|^2}\widetilde{Z}\in T^{-2}_\varphi E^2$. So $\sigma(\varphi)_{-2}(\xi) = \dot{X}$ is fulfiled. For $\dot{X} = E_{-1}(\dot{X})\in\mathfrak{g}_{-1}$ the vector field $\xi = \sum_{j=1}^n(\xi_X)_j\widetilde{X}_j + (\xi_Y)_j\widetilde{Y}_j \in T^{-1}_\varphi E^2$ with $\dot{X} \overset{!}{=} E_{-1}(aB^{-1}(\xi_X + i\xi_Y))$ satisfies $\sigma(\varphi)_{-1}(\xi) = \dot{X}$. In the same way we choose the vector field η matching $\dot{Y}\in\mathfrak{g}_{-2}$ or $\dot{Y}\in\mathfrak{g}_{-1}$. Then the torsion is given by

$$\begin{array}{rcl} t_{\hat{\varphi}}(\dot{X},\dot{Y}) & := & \begin{cases} pr_{\mathfrak{g}_{-2}\oplus\cdots\oplus\mathfrak{g}_{i+j+2}}\circ d(\sigma^*\hat{\theta}_{-2})_{\pi\hat{\varphi}}(\xi,\eta), & i+j<-2 \\ d(\sigma^*\hat{\theta}_{i+j})_{\pi\hat{\varphi}}(\xi,\eta), & i+j\geq -2 \end{cases} \\ & \in & \mathfrak{g}_{i+j}\oplus\cdots\oplus\mathfrak{g}_{i+j+2}. \end{array}$$

This gives the torsion of the elements $\sigma(\varphi) = \hat{\varphi}$.

- $\dot{X},\dot{Y}\in\mathfrak{g}_{-2}$

$$\begin{array}{rcl} t_{\hat{\varphi}}(\dot{X},\dot{Y}) & = & pr_{\mathfrak{g}_{-2}}\circ d(\sigma^*\hat{\theta}_{-2})_\varphi(\xi,\eta) \\ & = & -pr_{\mathfrak{g}_{-2}}\circ \underbrace{\hat{\varphi}_{-2}}_{=\sigma(\varphi)_{-2}=\theta^1_{-2}\oplus\theta^1_{-1}\oplus\omega_{\underline{G_0}}}\big([\xi,\eta]\big) \\ & = & -\theta^1_{-2}\big([\xi,\eta]\big) \\ & = & -\theta^1_{-2}\underbrace{\Big(\Big[-\frac{\dot{X}}{2|a|^2}\widetilde{Z},-\frac{\dot{Y}}{2|a|^2}\widetilde{Z}\Big]\Big)}_{=0} \\ & = & 0 \end{array}$$

- $\dot{X} \in \mathfrak{g}_{-2}$, $\dot{Y} \in \mathfrak{g}_{-1}$

$$\begin{aligned}
t_{\hat{\varphi}}(\dot{X},\dot{Y}) &= pr_{\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}} \circ d(\sigma^*\hat{\theta}_{-2})_{\varphi}(\xi,\eta) \\
&= -pr_{\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}} \circ \underbrace{\hat{\varphi}_{-2}}_{=\sigma(\varphi)_{-2}=\theta^1_{-2}\oplus\theta^1_{-1}\oplus\omega_{\underline{G_0}}} \big([\xi,\eta]\big) \\
&= -(\theta^1_{-2}\oplus\theta^1_{-1})\big([\xi,\eta]\big) \\
&= -(\theta^1_{-2}\oplus\theta^1_{-1})\big(\underbrace{\big[-\frac{\dot{X}}{2|a|^2}\widetilde{Z},\sum_{j=1}^{n}(\eta_X)_j\widetilde{X}_j+(\eta_Y)_j\widetilde{Y}_j\big]}_{=0}\big) \\
&= 0
\end{aligned}$$

- $\dot{X},\dot{Y} \in \mathfrak{g}_{-1}$

$$\begin{aligned}
t_{\hat{\varphi}}(\dot{X},\dot{Y}) &= d(\sigma^*\hat{\theta}_{-2})_{\varphi}(\xi,\eta) \\
&= -\hat{\theta}_{-2}\circ d\sigma\big([\xi,\eta]\big) \\
&= -\hat{\varphi}_{-2}\big([\xi,\eta]\big) \\
&= -\hat{\varphi}_{-2}\big(\big[\textstyle\sum(\xi_X)_j\widetilde{X}_j+(\xi_Y)_j\widetilde{Y}_j,\sum(\eta_X)_l\widetilde{X}_l+(\eta_Y)_l\widetilde{Y}_l\big]\big) \\
&= -\hat{\varphi}_{-2}\big(\textstyle\sum_j\big((\xi_X)_j(\eta_Y)_j-(\xi_Y)_j(\eta_X)_j\big)\widetilde{Z}\big) \\
&= -(\theta^1_{-2}\oplus\theta^1_{-1}\oplus\omega_{\underline{G_0}})\big(Im\langle\xi_x+i\xi_Y,\eta_X+i\eta_y\rangle\widetilde{Z}\big) \\
&= -\theta^1_{-2}\big(Im\langle\xi_x+i\xi_Y,\eta_X+i\eta_y\rangle\widetilde{Z}\big)\oplus\theta^1_{-1}(0)\oplus\omega_{\underline{G_0}}(0) \\
&\in \mathfrak{g}_{-2}
\end{aligned}$$

In the last case the homogeneous component of degree one vanishes.

Again the homogeneous component of degree one of the torsion of the elements $\sigma(\varphi)$ vanishes and we have a section into the next prolonged bundle E^3

$$\begin{aligned}
\sigma : E^2 &\longrightarrow E^3 \\
(\varphi_{-2},\varphi_{-1}) &\mapsto \big(\sigma(\varphi)_{-2},\sigma(\varphi)_{-1}\big) := \big(\theta^1_{-2}\oplus\theta^1_{-1}\oplus\omega_{\underline{G_0}},\theta^1_{-1}\oplus\omega_{\underline{P}/_{\underline{P}^2_+}}\big).
\end{aligned}$$

The section σ commutes with the right action of $\underline{P}/_{\underline{P}^2_+}$. I.e.

$$\begin{aligned}
E^3 &= He(n)\times\underline{P}/\underbrace{\underline{P}^3_+}_{=\{id\}} = He(n)\times\underline{P} \\
&= \left\{ \begin{array}{l} (x,g_0\exp(g_1)\exp(g_2)) \\ = R_{\exp(g_2)}\circ\sigma(x,g_0\exp(g_1)) \\ = R_{g_0\exp(g_1)\exp(g_2)}\circ\sigma(x,id) \end{array} \;\middle|\; x\in He(n), g_0\in\underline{G_0}, g_1\in\mathfrak{g}_1, g_2\in\mathfrak{g}_2 \right\}.
\end{aligned}$$

Right now there is no need to explicitly compute the frame form θ^3 of length three.
$(E^3,p^3,He(n);\theta^3)$ is the $\underline{P}$-frame bundle of degree three.
The bundle E^3 is already the $\underline{P}$-bundle we are looking for. However in order to obtain the normal Cartan connection two more prolongations are needed.

8.2.4 The Third and Forth Prolongation

We will now construct the third prolongation. The bundle $\hat{E}$ is now

$$\hat{E} := \left\{ \hat{\varphi} = (\hat{\varphi}_{-2},\hat{\varphi}_{-1}) \;\middle|\; \begin{array}{ll} \hat{\varphi}_{-2}: & T^{-2}E^3 \longrightarrow \mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1 \text{ an isomorphism with} \\ & pr_{\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}\oplus\mathfrak{g}_0}\circ\hat{\varphi}_{-2} = \theta^3_{-2} \text{ and } \hat{\varphi}_{-2}|_{T^{-1}E^3} = \theta^3_{-1} \\ \hat{\varphi}_{-1}: & T^{-1}E^3 \longrightarrow \mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1\oplus\mathfrak{g}_2 \text{ an isomorphism with} \\ & pr_{\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1}\circ\hat{\varphi}_{-1} = \theta^3_{-1} \text{ and } \hat{\varphi}_{-1}(\widetilde{A}) = A \\ & \text{for all } A\in\mathfrak{g}_0\oplus\mathfrak{g}_1\oplus\mathfrak{g}_2 \end{array} \right\}$$

with the global section

$$\begin{array}{rcl}\sigma : E^3 & \longrightarrow & \hat{E} \\ (\varphi_{-2}, \varphi_{-1}) & \mapsto & \big(\sigma(\varphi)_{-2}, \sigma(\varphi)_{-1}\big) := \big(\theta^1_{-2} \oplus \theta^1_{-1} \oplus \omega_{\underline{P}/_{\underline{P}^2_+}}, \theta^1_{-1} \oplus \omega_{\underline{P}}\big).\end{array}$$

Analogously to the prolongations above we find that the homogeneous component of degree one of the torsion of the elements $\sigma(\varphi)$ is zero. Thus

$$\begin{array}{rcl}\sigma : E^3 & \longrightarrow & \hat{E} \\ (\varphi_{-2}, \varphi_{-1}) & \mapsto & \big(\sigma(\varphi)_{-2}, \sigma(\varphi)_{-1}\big) := \big(\theta^1_{-2} \oplus \theta^1_{-1} \oplus \omega_{\underline{P}/_{\underline{P}^2_+}}, \theta^1_{-1} \oplus \omega_{\underline{P}}\big)\end{array}$$

is actually the unique section along which we have to pull back the frame form $\hat{\theta}$ in order to obtain the frame form θ^4 of degree four.
$(E^3, p^3, He(n); \theta^4)$ is the $\underline{P}$-frame bundle of degree four.
The last prolongation will now generate the normal Cartan connection.
The bundle

$$\hat{E} := \left\{ \hat{\varphi} = (\hat{\varphi}_{-2}, \theta^4_{-1}) \;\middle|\; \begin{array}{rl}\hat{\varphi}_{-2} : & T^{-2}E^3 \longrightarrow \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2 \text{ an isomorphism with} \\ & pr_{\mathfrak{g}_{-2}\oplus\cdots\oplus\mathfrak{g}_1} \circ \hat{\varphi}_{-2} = \theta^4_{-2} \text{ and } \hat{\varphi}_{-2}|_{T^{-1}E^3} = \theta^4_{-1}\end{array} \right\}$$

has the global section

$$\begin{array}{rcl}\sigma : E^3 & \longrightarrow & \hat{E} \\ (\varphi_{-2}, \varphi_{-1}) & \mapsto & \big(\sigma(\varphi)_{-2}, \sigma(\varphi)_{-1}\big) := \big(\theta^1_{-2} \oplus \theta^1_{-1} \oplus \omega_{\underline{P}}, \theta^4_{-1}\big).\end{array}$$

As above the homogeneous component of degree one of the torsion of the elements $\sigma(\varphi)$ vanishes and we have to pull back the frame form $\hat{\theta}$ along the section

$$\begin{array}{rcl}\sigma : E^3 & \longrightarrow & \hat{E} \\ (\varphi_{-2}, \varphi_{-1}) & \mapsto & \big(\sigma(\varphi)_{-2}, \sigma(\varphi)_{-1}\big) := \big(\theta^1_{-2} \oplus \theta^1_{-1} \oplus \omega_{\underline{P}}, \theta^4_{-1}\big).\end{array}$$

By this we obtain the frame form θ^5 of degree five. Since θ^5_{-1} contains no further information it suffices to compute θ^5_{-2} which is the Cartan connection.
We use the $\underline{P}$-equivariance and obtain for $\xi = \xi_{He} \oplus \xi_{\mathfrak{p}} = \dot{z}\widetilde{Z} \oplus \sum \dot{X}_j \widetilde{X}_j \oplus \sum \dot{Y}_j \widetilde{Y}_j \oplus \xi_{\mathfrak{p}}$

$$\begin{array}{rl} & (\theta^5_{-2})_{((X,Y,z),g)}(\xi) \\ \\ = & (\sigma^*\hat{\theta}_{-2})_{((X,Y,z),g)}(\xi) \\ = & (\hat{\theta}_{-2})_{\sigma((X,Y,z),g)} \circ d\sigma(\xi) \\ = & \sigma((X,Y,z),g)_{-2} \circ d\pi \circ d\sigma(\xi) \\ = & \big(\theta^1_{-2} \oplus \theta^1_{-1} \oplus \omega_{\underline{P}}\big)_{((X,Y,z),g)}(\xi_{He} \oplus \xi_{\mathfrak{p}}) \\ = & Ad(g^{-1}) \circ \big(\theta^1_{-2} \oplus \theta^1_{-1} \oplus \omega_{\underline{P}}\big)_{((X,Y,z),e)} \circ dR_{g^{-1}}(\xi_{He} \oplus \xi_{\mathfrak{p}}) \\ = & Ad(g^{-1}) \circ (\theta^1_{-2} \oplus \theta^1_{-1})_{((X,Y,z),e)} \circ dR_{g^{-1}}(\xi_{He}) + \omega_{\underline{P}}(\xi_{\mathfrak{p}}) \\ = & Ad(g^{-1})\big(E_{-2}\big(-2(\dot{z} - \sum_j x_j \dot{Y}_j)\big) \oplus E_{-1}(\dot{X} + i\dot{Y})\big) + dL_{g^{-1}}(\xi_{\mathfrak{p}}).\end{array}$$

We enlarge the Cartan bundle according to the construction of [CG08] by replacing the structure group $\underline{P}$ by P, which is quite simple in our case since the bundle is trivial. We obtain

$$\mathcal{G}_{He(n)} = He(n) \times P$$

joined by the Cartan connection

$$(\omega_{He(n)})_{\big((X,Y,z),p\big)}\big((\dot{X}, \dot{Y}, \dot{z}), \xi_{\mathfrak{p}}\big) = Ad(p^{-1})\Big(E_{-2}\big(-2(\dot{z}-\sum_j x_j \dot{Y}_j)\big) \oplus E_{-1}(\dot{X}+i\dot{Y})\Big) + dL_{p^{-1}}(\xi_{\mathfrak{p}}).$$

8.2.5 The Boundary

For a curve $\gamma = \underbrace{\begin{pmatrix} 1 & & \\ X+iY & I_n & \\ i\alpha - \frac{1}{2}\langle X+iY, X+iY\rangle & -X^t+iY^t & 1 \end{pmatrix}}_{\in P_-} \cdot \underbrace{g}_{\in P}$ in the homogeneous model we have for the Cartan connection applied to the derivative of this curve

$$\omega_{MC}(\dot\gamma) = Ad(g^{-1})\left(E_{-2}(\dot\alpha + \sum_j X_j\dot Y_j - \sum_j Y_j\dot X_j) \oplus E_{-1}(\dot X + i\dot Y)\right) + dL_{g^{-1}}\dot g.$$

Let us consider the map $\varphi : He(n) \longrightarrow G/_P$ of the Heisenberg group into the homogeneous model defined via

$$\begin{array}{rcl} \varphi : \quad He(n) & \longrightarrow & P_- := \left\{ \begin{pmatrix} 1 & & \\ V & I_n & \\ i\gamma - \frac{1}{2}\|V\|^2 & -V^* & 1 \end{pmatrix} \middle| \begin{array}{c} V \in \mathbb{C}^n \\ \gamma \in \mathbb{R} \end{array} \right\} \subset G/_P \\ (X,Y,z) & \mapsto & \varphi(X,Y,z) \\ := \begin{pmatrix} 1 & X^t & z \\ 0 & I_n & Y \\ 0 & 0 & 1 \end{pmatrix} & := & \begin{pmatrix} 1 & 0 & 0 \\ X+iY & I_n & 0 \\ i(-2z+\sum_j X_jY_j) - \frac{1}{2}\|X+iY\|^2 & -X^t+iY^t & 1 \end{pmatrix}. \end{array}$$

This map is covered by the P-principal bundle diffeomorphism

$$\begin{array}{rcl} \phi : \mathcal{G}_{He(n)} = He(n) \times P & \longrightarrow & \phi\big(\mathcal{G}_{He(n)}\big) \subset G \\ \big((X,Y,z),p\big) & \mapsto & R_p \circ \varphi(X,Y,z) \end{array}$$

and ϕ preserves the Cartan connections, $\phi^*\omega_G = \omega_{He(n)}$.

$$\begin{array}{rcl} \phi^*\omega_G(\dot\gamma) & = & \omega_G(d\phi \circ \dot\gamma) \\ & = & \omega_G\left(\frac{d}{dt}\phi((X,Y,z),g)\right) \\ & = & \omega_G\left(\frac{d}{dt}R_g\begin{pmatrix} 1 & 0 & 0 \\ X+iY & I_n & 0 \\ i(-2z+\sum_j X_jY_j) - \frac{1}{2}\|X+iY\|^2 & -X^t+iY^t & 1 \end{pmatrix}\right) \\ & = & Ad(g^{-1})\left(E_{-2}\left(\frac{d}{dt}(-2z+\sum_j X_jY_j) + \sum_j X_j\dot Y_j - \sum_j Y_j\dot X_j\right) \oplus E_{-1}(\dot X + i\dot Y)\right) \\ & & +dL_{g^{-1}}\dot g \\ & = & Ad(g^{-1})\left(E_{-2}\left(-2\dot z + 2\sum_j X_j\dot Y_j\right) \oplus E_{-1}(\dot X + i\dot Y)\right) + dL_{g^{-1}}\dot g \\ & = & Ad(g^{-1})\left(E_{-2}\big(-2(\dot z - \sum_j X_j\dot Y_j)\big) \oplus E_{-1}(\dot X + i\dot Y)\right) + dL_{g^{-1}}\dot g \\ & = & \omega_{He(n)}(\dot\gamma) \end{array}$$

Thus φ covered by ϕ is a geometric embedding of the Heisenberg group into the homogeneous model. The image of φ is the subgroup $P_- = \varphi\big(He(n)\big) \subsetneq G/_P$. And for ϕ we have $\phi(\mathcal{G}_{He(n)}) = P_- \cdot P \subsetneq G$. Hence the Cartan boundary of the Heisenberg group is exactly the topological boundary of $P_- \cdot P \subsetneq G$.

So in order to determine the Cartan boundary of the Heisenberg group we need to describe the group G and $P_- \cdot P \subsetneq G$. A matrix $A = \begin{pmatrix} a & X^* & b \\ Y & B & Z \\ c & W^* & d \end{pmatrix} \in M(\mathbb{C}, n+2)$ is an element of $SU(1, n+1) = G$ if its determinant is one and $A^*SA = S$.

$$\begin{aligned}
A^*SA &= A^* \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & X^* & b \\ Y & B & Z \\ c & W^* & d \end{pmatrix} \\
&= \begin{pmatrix} \overline{a} & Y^* & \overline{c} \\ X & B^* & W \\ \overline{b} & Z^* & \overline{d} \end{pmatrix} \cdot \begin{pmatrix} c & W^* & d \\ Y & B & Z \\ a & X^* & b \end{pmatrix} \\
&= \begin{pmatrix} \overline{a}c + a\overline{c} + \|Y\|^2 & * & * \\ cX + aW + B^*Y & XW^* + WX^* + B^*B & * \\ \overline{b}c + \overline{d}a + \langle Z, Y\rangle & \overline{b}W^* + \overline{d}X^* + Z^*B & \overline{b}d + b\overline{d} + \|Z\|^2 \end{pmatrix} \\
&\overset{!}{=} \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & 0 \end{pmatrix}
\end{aligned}$$

Thus we obtain the following equations

$$\begin{aligned}
I \quad 0 &= a\overline{c} + \overline{a}c + \|Y\|^2, \\
II \quad 0 &= b\overline{d} + \overline{b}d + \|Z\|^2, \\
III \quad 0 &= cX + aW + B^*Y, \\
IV \quad 0 &= dX + bW + B^*Z, \\
V \quad 1 &= a\overline{d} + \overline{b}c + \langle Z, Y\rangle, \\
VI \quad I_n &= B^*B + WX^* + XW^* \text{ and} \\
VII \quad 1 &= \det(A).
\end{aligned}$$

In Section 4.1 the subgroup $P_- \subset G$ was identified as

$$P_- = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ V & I_n & 0 \\ i\gamma - \frac{1}{2}\|V\|^2 & -V^* & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} V \in \mathbb{C}^n \\ \gamma \in \mathbb{R} \end{array} \right\}.$$

Lemma 8.1 *For* $A = \begin{pmatrix} a & X^* & b \\ Y & B & Z \\ c & W^* & d \end{pmatrix} \in G$ *and further* $a \neq 0$ *we have*

- $p_-(a, c, Y) := \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{a}Y & I_n & 0 \\ \frac{c}{a} & -\frac{1}{\overline{a}}Y^* & 1 \end{pmatrix} \in P_-$,
- $p(a, b, X, Y, Z, B) := \begin{pmatrix} a & X^* & b \\ 0 & B - \frac{1}{a}YX^* & Z - \frac{b}{a}Y \\ 0 & 0 & \overline{a}^{-1} \end{pmatrix} \in P$ *and*
- $p_-(a, c, Y) \cdot p(a, b, X, Y, Z, B) = A$.

Especially the image of the Cartan bundle of the Heisenberg group $\phi(\mathcal{G}_{He(n)})$ *is exactly given by the matricies of* G *with* $a \neq 0$.

Proof:

- For $p_-(a,c,Y)$ being an element of P_- the real part of $\frac{c}{a}$ has to be $-\frac{1}{2}\|\frac{1}{a}Y\|^2$. However according to equation I this is true:

$$\begin{aligned} -\tfrac{1}{2}\|\tfrac{1}{a}Y\|^2 &\overset{I}{=} \tfrac{1}{2a\overline{a}}(a\overline{c}+\overline{a}c) \\ &= \tfrac{1}{2}\big(\tfrac{\overline{c}}{\overline{a}}+\tfrac{c}{a}\big) \\ &= Re(\tfrac{c}{a}). \end{aligned}$$

- Several equations have to be checked when proving $p(a,b,X,Y,Z,B)\in P$:

$X^* = -a(Z-\frac{b}{a}Y)^*(B-\frac{1}{a}YX^*)$

$$\begin{aligned} & -a(Z-\tfrac{b}{a}Y)^*(B-\tfrac{1}{a}YX^*) \\ = \;& -aZ^*B+Z^*YX^*+\tfrac{a\overline{b}}{\overline{a}}Y^*B-\tfrac{\overline{b}}{\overline{a}}\|Y\|^2X^* \\ \overset{III,IV}{=} \;& a(\underline{\overline{b}W^*}+\overline{d}X^*)+\langle Z,Y\rangle X^*-\tfrac{a\overline{b}}{\overline{a}}(\underline{\overline{a}W^*}+\overline{c}X^*)-\tfrac{\overline{b}}{\overline{a}}\|Y\|^2X^* \\ \overset{I,V}{=} \;& a\overline{d}X^*+(1-a\overline{d}-\overline{b}c)X^*-\tfrac{a\overline{b}\overline{c}}{\overline{a}}X^*+\tfrac{\overline{b}}{\overline{a}}(a\overline{c}+\overline{a}c)X^* \\ = \;& X^*, \end{aligned}$$

$B-\frac{1}{a}YX^*\in U(n)$

$$\begin{aligned} & (B-\tfrac{1}{a}YX^*)^*\cdot(B-\tfrac{1}{a}YX^*) \\ = \;& B^*B-\tfrac{1}{\overline{a}}XY^*B-\tfrac{1}{a}B^*YX^*+\tfrac{1}{|a|^2}\|Y\|^2XX^* \\ \overset{III}{=} \;& B^*B+\tfrac{1}{\overline{a}}X(\overline{a}W^*+\overline{c}X^*)+\tfrac{1}{a}(aW+cX)X^*+\tfrac{1}{|a|^2}\|Y\|^2XX^* \\ = \;& \underbrace{B^*B+XW^*+WX^*}_{\overset{VI}{=}I_n}+\underbrace{(\frac{\overline{c}}{\overline{a}}+\frac{c}{a}+\frac{\|Y\|^2}{|a|^2})}_{\overset{I}{=}0}XX^* \\ = \;& I_n, \end{aligned}$$

$\frac{b}{a}+\frac{\overline{b}}{\overline{a}}+\|Z-\frac{b}{a}Y\|^2=0$

$$\begin{aligned} & \tfrac{b}{a}+\tfrac{\overline{b}}{\overline{a}}+\|Z-\tfrac{b}{a}Y\|^2 \\ = \;& \tfrac{b}{a}+\tfrac{\overline{b}}{\overline{a}}+\underbrace{\|Z\|^2}_{\overset{II}{=}-b\overline{d}-\overline{b}d}+\tfrac{b\overline{b}}{a\overline{a}}\underbrace{\|Y\|^2}_{\overset{I}{=}-a\overline{c}-\overline{a}c}-\tfrac{b}{a}\underbrace{\langle Z,Y\rangle}_{\overset{V}{=}1-a\overline{d}-\overline{b}c}-\tfrac{\overline{b}}{\overline{a}}\underbrace{\langle Y,Z\rangle}_{\overset{V}{=}1-\overline{a}d-b\overline{c}} \\ = \;& \tfrac{b}{a}+\tfrac{\overline{b}}{\overline{a}}-b\overline{d}-\overline{b}d-\tfrac{b\overline{b}\overline{c}}{\overline{a}}-\tfrac{b\overline{b}c}{a}-\tfrac{b}{a}+b\overline{d}+\tfrac{b\overline{b}c}{a}-\tfrac{\overline{b}}{\overline{a}}+\overline{b}d+\tfrac{b\overline{b}\overline{c}}{\overline{a}} \\ = \;& 0 \text{ and} \end{aligned}$$

$\det\big(p(a,b,X,Y,Z,B)\big)=1.$

Once we have proven that $p_-(a,c,Y)\cdot p(a,b,X,Y,Z,B)=A$ we have for the determinante

$$\underbrace{\det(A)}_{=1}=\underbrace{\det(p_-(a,c,Y))}_{=1}\cdot\det(p(a,b,X,Y,Z,B)).$$

Thus the determinante of $p(a,b,X,Y,Z,B)$ has to be one as well.

- The product of $p_-(a,c,Y)$ and $p(a,b,X,Y,Z,B)$ is

$$\begin{aligned} & p_-(a,c,Y)\cdot p(a,b,X,Y,Z,B) \\ = & \begin{pmatrix} a & X^* & b \\ Y & B & Z \\ c & \frac{c}{a}X^*-\frac{1}{\overline{a}}Y^*B+\frac{1}{|a|^2}\|Y\|^2X^* & \frac{bc}{a}-\frac{1}{\overline{a}}\langle Y,Z\rangle+\frac{b}{|a|^2}\|Y\|^2+\frac{1}{\overline{a}} \end{pmatrix}. \end{aligned}$$

This is exactly the matrix A since

$W^* = \frac{c}{a}X^* - \frac{1}{\overline{a}}Y^*B + \frac{1}{|a|^2}\|Y\|^2X^*$

$$\begin{array}{rl} & \frac{c}{a}X^* - \frac{1}{\overline{a}}Y^*B + \frac{1}{|a|^2}\|Y\|^2X^* \\ \overset{I,III}{=} & \frac{c}{a}X^* - \frac{1}{\overline{a}}(\overline{a}W^* + \overline{c}X^*) - \frac{1}{|a|^2}(a\overline{c} + \overline{a}c)X^* \\ = & W^* \text{ and} \end{array}$$

$d = \frac{bc}{a} - \frac{1}{\overline{a}}\langle Y, Z\rangle + \frac{b}{|a|^2}\|Y\|^2 + \overline{a}^{-1}$

$$\begin{array}{rl} & \frac{bc}{a} - \frac{1}{\overline{a}}\langle Y, Z\rangle + \frac{b}{|a|^2}\|Y\|^2 + \overline{a}^{-1} \\ \overset{V,I}{=} & \frac{bc}{a} - \frac{1}{\overline{a}}(1 - \overline{a}d - b\overline{c}) - \frac{b}{|a|^2}(a\overline{c} + \overline{a}c) + \overline{a}^{-1} \\ = & d. \end{array}$$

Thus all statements of the lemma are proven and we obtain that the image of the Cartan bundle of the Heisenberg group $\phi(\mathcal{G}_{He(n)}) = P_- \cdot P$ is exactly given by the matricies of G with $a \neq 0$.

$\square$

From the lemma above we also see that $P_- \cdot P \subset G$ is a dense subset. Its boundary is given by all matricies of G with a zero in the first line of the first row.

I.e. the boundary is given by all matricies $\begin{pmatrix} a & X^* & b \\ Y & B & Z \\ c & W^* & d \end{pmatrix}$ with determinante one, $a = 0$ and

$$\begin{array}{rrcl} I & 0 & = & \|Y\|^2, \\ II & 0 & = & b\overline{d} + \overline{b}d + \|Z\|^2, \\ III & 0 & = & cX + B^*Y, \\ IV & 0 & = & dX + bW + B^*Z, \\ V & 1 & = & \overline{b}c + \langle Z, Y\rangle \text{ and} \\ VI & I_n & = & B^*B + WX^* + XW^*. \end{array}$$

Thus according to equation I we have $Y = 0$ which gives

$$\begin{array}{rrcl} II & 0 & = & b\overline{d} + \overline{b}d + \|Z\|^2, \\ III & 0 & = & cX, \\ IV & 0 & = & dX + bW + B^*Z, \\ V & 1 & = & \overline{b}c \text{ and} \\ VI & I_n & = & B^*B + WX^* + XW^*. \end{array}$$

With equation V we know that $c = \frac{1}{\overline{b}}$ does not vanish and consequently X has to vanish in order to fulfil equation III.

$$\begin{array}{rrcl} II & 0 & = & \frac{\overline{d}}{\overline{c}} + \frac{d}{c} + \|Z\|^2, \\ IV & 0 & = & \frac{1}{\overline{c}}W + B^*Z \text{ and} \\ VI & I_n & = & B^*B. \end{array}$$

Hence we obtain for the Cartan boundary of the Cartan bundle of the Heisenberg group:

$$\begin{aligned}
\partial_{CB}\mathcal{G}_{He(n)} &= \partial(P_- \cdot P)\\
&= \left\{ \begin{pmatrix} 0 & 0 & \frac{1}{\bar{c}} \\ 0 & B & Z \\ c & -cZ^*B & d \end{pmatrix} \;\middle|\; \begin{array}{l} c \in \mathbb{C}^*, d \in \mathbb{C}, B \in U(n), Z \in \mathbb{C}^n \\ \frac{\bar{d}}{\bar{c}} + \frac{d}{c} + \|Z\|^2 = 0 \\ -\frac{\bar{c}}{c}\det B = 1 \end{array} \right\}\\
&= \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c & -cZ^*B & d \\ 0 & B & Z \\ 0 & 0 & \frac{1}{\bar{c}} \end{pmatrix} \;\middle|\; \begin{array}{l} c \in \mathbb{C}^*, d \in \mathbb{C}, B \in U(n), Z \in \mathbb{C}^n \\ \frac{\bar{d}}{\bar{c}} + \frac{d}{c} + \|Z\|^2 = 0 \\ \frac{\bar{c}}{c}\det B = 1 \end{array} \right\}\\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot P\\
&=: \hat{I} \cdot P.
\end{aligned}$$

With the last representation of the Cartan boundary of the Cartan bundle of the Heisenberg group we can easily determine the Cartan boundary of the Heisenberg group itself,

$$\partial_{CB} He(n) = \{\text{point}\}.$$

8.3 The Cartan Boundary of the Fefferman Space of the Heisenberg Group

Now we use the construction of [CG08] to determine the structure of the Cartan boundary of the Fefferman space of the Heisenberg group. The Cartan bundel of the Fefferman space is given as

$$\begin{aligned}
\mathcal{G}_{\mathcal{F}} &= \mathcal{G}_{He(n)} \times_{G\cap\widetilde{P}} \widetilde{P}\\
&\simeq P_- \cdot P \times_{G\cap\widetilde{P}} \widetilde{P}.
\end{aligned}$$

The Cartan connection of this bundle is defined to make this a geometric embedding since the embedding $\mathcal{G}_{He(n)} \longrightarrow G$ is geometric.

$$\begin{array}{ccccc}
\mathcal{G}_{\mathcal{F}} = \mathcal{G}_{He(n)} \times_{G\cap\widetilde{P}} \widetilde{P} & \longrightarrow & P_- \cdot P \times_{G\cap\widetilde{P}} \widetilde{P} & \subset & \widetilde{G} = G \times_{G\cap\widetilde{P}} \widetilde{P}\\
(\omega_{\mathcal{F}})_{[u,\widetilde{p}]} & & & & \omega_{\widetilde{G}}\\
= Ad(\widetilde{p}^{-1}) \circ \pi^*_{\mathcal{G}_{He(n)}} \omega_{He(n)} + \pi^*_{\widetilde{P}}\omega_{\widetilde{P}} & & & & = Ad(\widetilde{p}^{-1}) \circ \pi^*_G \omega_G + \pi^*_{\widetilde{P}}\omega_{\widetilde{P}}
\end{array}$$

The identifications $\widetilde{G} = G \times_{G\cap\widetilde{P}} \widetilde{P}$ and $\omega_{\widetilde{G}} = Ad(\widetilde{p}^{-1}) \circ \pi^*_G \omega_G + \pi^*_{\widetilde{P}}\omega_{\widetilde{P}}$ are obtained from the homogeneous model in Sections 5.1 and 5.2. We continue

$$\begin{aligned}
\mathcal{G}_{\mathcal{F}} &\simeq P_- \cdot P \times_{G\cap\widetilde{P}} \widetilde{P}\\
&= G \setminus (\hat{I} \cdot P) \times_{G\cap\widetilde{P}} \widetilde{P}\\
&= \widetilde{G} \setminus (\hat{I} \cdot P \times_{G\cap\widetilde{P}} \widetilde{P}).
\end{aligned}$$

As we have seen in Section 5.4 we have $P = S^1 \ltimes (\widetilde{P}^+ \cap G)$ and $P/_{\widetilde{P}\cap G}$ gives exactly the fibre of the Fefferman space in this construction. Consequently we can write

$$\begin{aligned}
\mathcal{G}_{\mathcal{F}} &= \widetilde{G} \setminus (\hat{I} \cdot P \times_{G\cap\widetilde{P}} \widetilde{P})\\
&= \widetilde{G} \setminus (\hat{I} \cdot S^1 \times \widetilde{P}).
\end{aligned}$$

This is a dense subset of the whole group $\widetilde{G}$ and so the Cartan boundary of the Fefferman space of the Heisenberggroup is

$$\begin{aligned}
\partial_{CB}\mathcal{G}_{\mathcal{F}} &\simeq \hat{I} \cdot S^1 \times \widetilde{P} \text{ and}\\
\partial_{CB}\mathcal{F} &\simeq S^1.
\end{aligned}$$

From the considerations above we also see that the completed Fefferman space of the Heisenberg group is a manifold and a S^1-principal bundle over the completed Heisenberg group.

$$\begin{tikzcd}
S^1 \circlearrowleft \mathcal{F} \arrow[r] \arrow[d, "\pi"'] & \overline{\mathcal{F}} \simeq \mathcal{F}\dot{\cup}S^1 \circlearrowright S^1 \arrow[d, "\overline{\pi}"] \\
He(n) \arrow[r] & \overline{He(n)} \simeq He(n)\dot{\cup}\{point\}
\end{tikzcd}$$

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